THE TIME INVERSION FOR MODIFIED OSCILLATORS

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Abstract. We discuss a new completely integrable case of the time-dependent Schrödinger equation in $\boldsymbol{R}^n$ with variable coefficients for a modified oscillator, which is dual with respect to the time inversion to a model of the quantum oscillator recently considered by Meiler, Cordero-Soto, and Suslov. A second pair of dual Hamiltonians is found in the momentum representation. Our examples show that in mathematical physics and quantum mechanics a change in the direction of time may require a total change of the system dynamics in order to return the system back to its original quantum state. Particular solutions of the corresponding nonlinear Schrödinger equations are obtained. A Hamiltonian structure of the classical integrable problem and its quantization are also discussed.

1. Introduction

The Cauchy initial value problem for the Schrödinger equation

$$i\frac{\partial\psi}{\partial t}=H\left(t\right)\psi \tag{1.1}$$

for a certain modified oscillator is explicitly solved in Ref. [40] for the case of n dimensions in $\boldsymbol{R}^n$. When $n=1$ the Hamiltonian considered by Meiler, Cordero-Soto, and Suslov has the form

$$H\left(t\right)=\frac{1}{2}\left(aa^{\dagger}+a^{\dagger}a\right)+\frac{1}{2}e^{2it}a^{2}+\frac{1}{2}e^{-2it}\left(a^{\dagger}\right)^{2}, \tag{1.2}$$

where the creation and annihilation operators are defined as in [28]:

$$a^{\dagger}=\frac{1}{i\sqrt{2}}\left(\frac{\partial}{\partial x}-x\right),\qquad a=\frac{1}{i\sqrt{2}}\left(\frac{\partial}{\partial x}+x\right). \tag{1.3}$$

The corresponding time evolution operator is found in [40] as an integral operator

$$\psi\left(x,t\right)=U\left(t\right)\psi\left(x,0\right)=\int_{-\infty}^{\infty}G\left(x,y,t\right)\ \psi\left(y,0\right)\ dy \tag{1.4}$$

with the kernel (Green's function or propagator) given in terms of trigonometric and hyperbolic functions as follows

$$\begin{aligned}G\left(x,y,t\right) &= \frac{1}{\sqrt{2\pi i\left(\cos t\sinh t+\sin t\cosh t\right)}}\\ &\quad\times\exp\left(\frac{\left(x^{2}-y^{2}\right)\sin t\sinh t+2xy-\left(x^{2}+y^{2}\right)\cos t\cosh t}{2i\left(\cos t\sinh t+\sin t\cosh t\right)}\right).\end{aligned} \tag{1.5}$$

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It is worth noting that the time evolution operator is known explicitly only in a few special cases. An important example of this source is the forced harmonic oscillator originally considered by Richard Feynman in his path integrals approach to the nonrelativistic quantum mechanics [23], [24], [25], [26], and [27]; see also [37]. Since then this problem and its special and limiting cases were discussed by many authors; see Refs. [5], [29], [32], [39], [42], [65] for the simple harmonic oscillator and Refs. [2], [8], [31], [45], [53] for the particle in a constant external field and references therein. Furthermore, in Ref. [20] the time evolution operator for the one-dimensional Schrödinger equation (1.1) has been constructed in a general case when the Hamiltonian is an arbitrary quadratic form of the operator of coordinate and the operator of linear momentum. In this approach, the above mentioned exactly solvable models, including the modified oscillator of [40], are classified in terms of elementary solutions of a certain characteristic equation related to the Riccati differential equation.

In the present paper we find the time evolution operator for a "dual" time-dependent Schrödinger equation of the form

$$i\frac{\partial\psi}{\partial\tau} = H\left(\tau\right)\psi, \qquad \tau = \frac{1}{2}\sinh\left(2t\right) \tag{1.6}$$

with another "exotic" Hamiltonian of a modified oscillator given by

$$H\left(\tau\right) = \frac{1}{2}\left(aa^{\dagger} + a^{\dagger}a\right) + \frac{1}{2}e^{-i\arctan(2\tau)}a^{2} + \frac{1}{2}e^{i\arctan(2\tau)}\left(a^{\dagger}\right)^{2}. \tag{1.7}$$

We show that the corresponding propagator can be obtain from expression (1.5) by interchanging the coordinates $x \leftrightarrow y$. This implies that these two models are related to each other with respect to the inversion of time, which is the main result of this article.

The paper is organized as follows. In section 2 we derive the propagators for the Hamiltonians (1.2) and (1.7) following the method of [20] — expression (1.5) was obtain in [40] by a totally different approach using $SU(1,1)$-symmetry of the n-dimensional oscillator wave functions and the Meixner–Pollaczek polynomials. Another pair of completely integrable dual Hamiltonians is also discussed here. The "hidden" symmetry of quadratic propagators is revealed in section 3. The next section is concerned with the complex form of the propagators, which unifies Green's functions for several classical models by geometric means. In section 5 we consider the inverses of the corresponding time evolution operators and its relation with the inversion of time. A transition to the momentum representation in section 6 gives the reader a new insight on the symmetries of the quadratic Hamiltonians under consideration together with a set of identities for the corresponding time evolution operators. The n-dimensional case is discussed in sections 7 and 8. Particular solutions of the corresponding nonlinear Schrödinger equations are constructed in section 9. The last section is concerned with the ill-posedness of the Schrödinger equations. Three Appendixes at the end of the paper deal with required solutions of a certain type of characteristic equations, a quantum Hamiltonian transformation, and a Hamiltonian structure of the characteristic equations under consideration, respectively.

As in [20], [40] and [61], we are dealing here with solutions of the time-dependent Schrödinger equation with variable coefficients. The case of a corresponding diffusion-type equation is investigated in [62]. These exactly solvable models are of interest in a general treatment of the nonlinear evolution equations; see [9], [14], [18], [19], [22], [34], [41], [64] and [3], [10], [11], [12], [13], [15], [16], [33], [44], [50], [54], [55], [57], [60] and references therein. They facilitate, for instance, a detailed study of problems related to global existence and uniqueness of solutions for the nonlinear Schrödinger equations with general quadratic Hamiltonians. Moreover, these explicit solutions

can also be useful when testing numerical methods of solving the Schrödinger and diffusion-type equations with variable coefficients.

2. Derivation of The Propagators

The fundamental solution of the time-dependent Schrödinger equation with the quadratic Hamiltonian of the form

$$i\frac{\partial \psi}{\partial t}=-a\left(t\right)\frac{\partial^{2}\psi}{\partial x^{2}}+b\left(t\right)x^{2}\psi-i\left(c\left(t\right)x\frac{\partial \psi}{\partial x}+d\left(t\right)\psi\right) \tag{2.1}$$

in two interesting special cases, namely,

$$a=\cos^{2}t,\quad b=\sin^{2}t,\quad c=2d=\sin\left(2t\right) \tag{2.2}$$

and

$$a=\cosh^{2}t,\quad b=\sinh^{2}t,\quad c=2d=-\sinh\left(2t\right), \tag{2.3}$$

corresponding to the Hamiltonians (1.2) and (1.7), respectively, (we give details of the proof in the Appendix B), can be found by the method proposed in [20] in the form

$$\psi=G\left(x,y,t\right)=\frac{1}{\sqrt{2\pi i\mu\left(t\right)}}\ e^{i\left(\alpha(t)x^{2}+\beta(t)xy+\gamma(t)y^{2}\right)}, \tag{2.4}$$

where

$$\alpha\left(t\right)=\frac{1}{4a\left(t\right)}\frac{\mu'\left(t\right)}{\mu\left(t\right)}-\frac{d\left(t\right)}{2a\left(t\right)}, \tag{2.5}$$

$$\beta\left(t\right)=-\frac{1}{\mu\left(t\right)},\qquad \frac{d\gamma}{dt}+\frac{a\left(t\right)}{\mu\left(t\right)^{2}}=0, \tag{2.6}$$

$$\gamma\left(t\right)=\frac{a\left(t\right)}{\mu\left(t\right)\mu'\left(t\right)}+\frac{d\left(0\right)}{2a\left(0\right)}-4\int_{0}^{t}\frac{a\left(\tau\right)\sigma\left(\tau\right)}{\left(\mu'\left(\tau\right)\right)^{2}}\ d\tau, \tag{2.7}$$

and the function $\mu\left(t\right)$ satisfies the characteristic equation

$$\mu''-\tau\left(t\right)\mu'+4\sigma\left(t\right)\mu=0 \tag{2.8}$$

with

$$\tau\left(t\right)=\frac{a'}{a}-2c+4d,\qquad \sigma\left(t\right)=ab-cd+d^{2}+\frac{d}{2}\left(\frac{a'}{a}-\frac{d'}{d}\right) \tag{2.9}$$

subject to the initial data

$$\mu\left(0\right)=0,\qquad \mu'\left(0\right)=2a\left(0\right)\neq 0. \tag{2.10}$$

Equation (2.5) (more details can be found in [20]) allows us to integrate the familiar Riccati nonlinear differential equation emerging when one substitutes (2.4) into (2.1). See, for example, [30], [43], [51], [52], [68] and references therein. A Hamiltonian structure of these characteristic equations is discussed in Appendix C.

In the case (2.2), the characteristic equation has a special form of Ince's equation [38]

$$\mu''+2\tan t\ \mu'-2\mu=0. \tag{2.11}$$

Two linearly independent solutions are found in [20]:

$$\begin{aligned}\mu_{1}&=\cos t\cosh t+\sin t\sinh t=W\left(\cos t,\sinh t\right), &(2.12)\\ \mu_{2}&=\cos t\sinh t+\sin t\cosh t=W\left(\cos t,\cosh t\right) &(2.13)\end{aligned}$$

with the Wronskian $W(\mu_1, \mu_2) = 2\cos^2 t = 2a$. Another method of integration of all characteristic equations from this section is discussed in the Appendix A; see Table 1 at the end of the paper for the sets of fundamental solutions. The second case (2.3) gives

$$\mu'' - 2\tanh t\ \mu' + 2\mu = 0 \tag{2.14}$$

and the two linearly independent solutions are [62]

$$\mu_2 = \cos t \sinh t + \sin t \cosh t = W(\cos t, \cosh t), \tag{2.15}$$

$$\mu_3 = \sin t \sinh t - \cos t \cosh t = W(\sin t, \cosh t) \tag{2.16}$$

with $W(\mu_2, \mu_3) = 2\cosh^2 t = 2a$. Equation (2.14) can be obtain from (2.11) as a result of the substitution $t \to it$. Also, $W(\mu_1, \mu_3) = \sin(2t) + \sinh(2t)$. The common solution of the both characteristic equations, namely,

$$\mu(t) = \mu_2 = \cos t \sinh t + \sin t \cosh t, \tag{2.17}$$

satisfies the initial conditions (2.10).

From (2.4)–(2.7), as a result of elementary calculations, one arrives at the Green function (1.5) in the case (2.2) and has to interchange there $x \leftrightarrow y$ in the second case (2.3). The reader can see some calculation details in section 9, where more general solutions are found in a similar way. The next section explains this unusual symmetry between two propagators from a more general point of view.

Two more completely integrable cases of the dual quadratic Hamiltonians occur when

$$a = \sin^2 t, \quad b = \cos^2 t, \quad c = 2d = -\sin(2t) \tag{2.18}$$

and

$$a = \sinh^2 t, \quad b = \cosh^2 t, \quad c = 2d = \sinh(2t). \tag{2.19}$$

The corresponding characteristic equations are

$$\mu'' - 2\cot t\ \mu' - 2\mu = 0 \tag{2.20}$$

and

$$\mu'' - 2\coth t\ \mu' + 2\mu = 0, \tag{2.21}$$

respectively, with a common solution

$$\mu(t) = \mu_4 = \sin t \cosh t - \cos t \sinh t = W(\sin t, \sinh t) \tag{2.22}$$

such that $\mu(0) = \mu'(0) = \mu''(0) = 0$ and $\mu'''(0) = 4$. Once again, from (2.4)–(2.7) one arrives at the Green function

$$\begin{aligned} G(x,y,t) &= \frac{1}{\sqrt{2\pi i (\sin t \cosh t - \cos t \sinh t)}} \\ &\quad \times \exp\left(\frac{(x^2+y^2)\cos t \cosh t - 2xy + (x^2-y^2)\sin t \sinh t}{2i(\cos t \sinh t - \sin t \cosh t)}\right) \end{aligned} \tag{2.23}$$

in the case (2.18) and has to interchange there $x \leftrightarrow y$ in the second case (2.19). The corresponding asymptotic formula takes the form

$$G(x,y,t) = \frac{e^{i\left(\alpha(t)x^2 + \beta(t)xy + \gamma(t)y^2\right)}}{\sqrt{2\pi i \mu(t)}} \sim \frac{1}{\sqrt{4\pi i \varepsilon}} \exp\left(i\frac{(x-y)^2}{4\varepsilon}\right) \tag{2.24}$$

as $\varepsilon = t^3/3 \to 0^+$. We will show in section 6, see Eqs. (6.1) and (6.8), that our cases (2.2)–(2.3) and (2.18)–(2.19) are related to each other by means of the Fourier transform.

We have considered some elementary solutions of the characteristic equation (2.8), which are of interest in this paper. Generalizations to the forced modified oscillators are obvious; see Ref. [40]. More complicated cases may include special functions, like Bessel, hypergeometric or elliptic functions [1], [20], [47], [49], [63], and [68].

3. On a "Hidden" Symmetry of Quadratic Propagators

Here we shall elaborate on the symmetry of propagators with respect to the substitution $x \leftrightarrow y$.

Lemma 1. *Consider two time-dependent Schrödinger equations with quadratic Hamiltonians*

$$i\frac{\partial \psi}{\partial t} = -a_k(t)\frac{\partial^2 \psi}{\partial x^2} + b_k(t)\, x^2\psi - i\left(c_k(t)\, x\frac{\partial \psi}{\partial x} + d_k(t)\,\psi\right) \quad (k=1,2), \tag{3.1}$$

where $c_1 - 2d_1 = c_2 - 2d_2 = \varepsilon(t)$ and $d_k(0) = 0$. Suppose that the initial value problems for corresponding characteristic equations

$$\mu'' - \tau_k(t)\,\mu' + 4\sigma_k(t)\,\mu = 0, \qquad \mu(0) = 0, \quad \mu'(0) = 2a_k(0) \neq 0 \tag{3.2}$$

with

$$\tau_k(t) = \frac{a_k'}{a_k} - 2c_k + 4d_k, \quad \sigma_k(t) = a_kb_k - c_kd_k + d_k^2 + \frac{d_k}{2}\left(\frac{a_k'}{a_k} - \frac{d_k'}{d_k}\right) \tag{3.3}$$

have a joint solution $\mu(t)$ and, in addition, the following relations hold

$$4\left(a_1b_1 - c_1d_1 + d_1^2\right) = \frac{4a_1a_2h^2 - (\mu')^2}{\mu^2} - 2\varepsilon\frac{\mu'}{\mu} = 4\left(a_2b_2 - c_2d_2 + d_2^2\right), \tag{3.4}$$

where $h(t) = \exp\left(-\int_0^t \varepsilon(\tau)\ d\tau\right)$. Then the corresponding fundamental solutions

$$\psi_k = G_k(x,y,t) = \frac{1}{\sqrt{2\pi i\mu(t)}}\ e^{i\left(\alpha_k(t)x^2 + \beta_k(t)xy + \gamma_k(t)y^2\right)} \tag{3.5}$$

possess the following symmetry

$$\alpha(t) = \alpha_1(t) = \gamma_2(t), \quad \gamma(t) = \gamma_1(t) = \alpha_2(t), \quad \beta(t) = \beta_1(t) = \beta_2(t) \tag{3.6}$$

and

$$G_1(x,y,t) = G_2(y,x,t). \tag{3.7}$$

This property holds for a single Schrödinger equation under the single hypothesis (3.4).

Indeed, according to Ref. [20],

$$\beta(t) = \beta_1(t) = \beta_2(t) = -\frac{h(t)}{\mu(t)} \tag{3.8}$$

in the case of a joint solution $\mu(t)$ of two characteristic equations. In view of the structure of propagators for general quadratic Hamiltonians found in [20], the symmetry under consideration holds if we have

$$\alpha = \frac{1}{4a_1}\frac{\mu'}{\mu} - \frac{d_1}{2a_1}, \qquad \frac{d\alpha}{dt} + a_2\frac{h^2}{\mu^2} = 0 \tag{3.9}$$

and

$$\gamma = \frac{1}{4a_2}\frac{\mu'}{\mu} - \frac{d_2}{2a_2}, \qquad \frac{d\gamma}{dt} + a_1\frac{h^2}{\mu^2} = 0, \tag{3.10}$$

simultaneously. Excluding α from (3.9), one gets

$$\mu'' - \frac{a_1'}{a_1}\mu' + 2d_1\left(\frac{a_1'}{a_1} - \frac{d_1'}{d_1}\right)\mu = \frac{(\mu')^2 - 4a_1a_2h^2}{\mu}. \tag{3.11}$$

Comparison with the characteristic equation results in the first condition in (3.4). The case of γ, which gives the second condition, is similar. This completes the proof.

A few examples are in order. When $a = 1/2$, $b = c = d = 0$, and $\mu'' = 0$, $\mu = t$, one gets

$$G(x,y,t) = \frac{1}{\sqrt{2\pi i t}}\ \exp\left(\frac{i(x-y)^2}{2t}\right) \tag{3.12}$$

as the free particle propagator [27] with an obvious symmetry under consideration. Our criteria (3.4), namely, $4a^2 = (\mu')^2$, stands.

The simple harmonic oscillator with $a = b = 1/2$, $c = d = 0$ and $\mu'' + \mu = 0$, $\mu = \sin t$ has the familiar propagator of the form

$$G(x,y,t) = \frac{1}{\sqrt{2\pi i \sin t}}\ \exp\left(\frac{i}{2\sin t}\left(\left(x^2+y^2\right)\cos t - 2xy\right)\right), \tag{3.13}$$

which is studied in detail at [5], [29], [32], [39], [42], [65]. (For an extension to the case of the forced harmonic oscillator including an extra velocity-dependent term and a time-dependent frequency, see [23], [24], [27] and [37].) Our condition (3.4) takes the form of the trigonometric identity

$$4ab = \frac{4a^2 - (\mu')^2}{\mu^2}, \tag{3.14}$$

which confirms the symmetry of the propagator.

For the quantum damped oscillator [21] $a = b = \omega_0/2$, $c = d = -\lambda$ and

$$\begin{aligned} G(x,y,t) &= \sqrt{\frac{\omega e^{\lambda t}}{2\pi i \omega_0 \sin \omega t}}\ \exp\left(\frac{i\omega}{2\omega_0 \sin\omega t}\left(\left(x^2+y^2\right)\cos\omega t - 2xy\right)\right) \\ &\quad \times \exp\left(\frac{i\lambda}{2\omega_0}\left(x^2-y^2\right)\right) \end{aligned} \tag{3.15}$$

with $\omega = \sqrt{\omega_0^2 - \lambda^2} > 0$ and $\mu = (\omega_0/\omega)\, e^{-\lambda t}\sin\omega t$. The criterion

$$4ab = \frac{4(ah)^2 - (\mu')^2}{\mu^2} - 2\varepsilon\frac{\mu'}{\mu}, \tag{3.16}$$

where $\varepsilon = c - 2d = \lambda$ and $h = e^{-\lambda t}$, holds. But here $d(0) = -\lambda \neq 0$ and a more detailed analysis of asymptotics gives an extra antisymmetric term in the propagator above; see [21] for more details.

The case of Hamiltonians (1.2) and (1.7) corresponds to

$$a_1 = \cos^2 t, \quad b_1 = \sin^2 t, \quad c_1 = 2d_1 = \sin(2t) \tag{3.17}$$

and

$$a_2 = \cosh^2 t, \quad b_2 = \sinh^2 t, \quad c_2 = 2d_2 = -\sinh(2t), \tag{3.18}$$

respectively. Our criteria (3.4) are satisfied in view of an obvious identity

$$4a_1a_2 = (\mu')^2 . \tag{3.19}$$

The characteristic function is given by (2.17). This explains the propagator symmetry found in the previous section.

Our last dual pair of quadratic Hamiltonians has the following coefficients

$$a_1 = \sin^2 t, \quad b_1 = \cos^2 t, \quad c_1 = 2d_1 = -\sin(2t) \tag{3.20}$$

and

$$a_2 = \sinh^2 t, \quad b_2 = \cosh^2 t, \quad c_2 = 2d_2 = \sinh(2t) . \tag{3.21}$$

The criteria (3.4) are satisfied in view of the identity (3.19) with the characteristic function (2.22) and, therefore, the propagator (2.23) obeys the symmetry under the substitution $x \leftrightarrow y$.

Remark 1. *A simple relation*

$$\frac{\mu'}{\mu} = 4\frac{\sigma_1 - \sigma_2}{\tau_1 - \tau_2}, \tag{3.22}$$

which is valid for a joint solution of two characteristic equations, can be used in our criteria (3.4).

Although we have formulated the hypotheses of our lemma for the Green functions only, it can be applied to solutions with regular initial data. For instance, a pair of characteristic equations (2.11) and (2.21) has a joint solution given by (2.12), which does not satisfy initial conditions required for the Green functions. The coefficients of the corresponding quadratic Hamiltonians are

$$a_1 = \cos^2 t, \quad b_1 = \sin^2 t, \quad c_1 = 2d_1 = \sin(2t) \tag{3.23}$$

and

$$a_2 = \sinh^2 t, \quad b_2 = \cosh^2 t, \quad c_2 = 2d_2 = \sinh(2t) . \tag{3.24}$$

The criteria (3.4) are satisfied in view of the identity (3.19) and the particular solution

$$\begin{aligned} \psi = K(x,y,t) = {} & \frac{1}{\sqrt{2\pi(\cos t\cosh t + \sin t\sinh t)}} \\ & \times \exp\left(\frac{(x^2+y^2)\sin t\cosh t - 2xy - (x^2-y^2)\cos t\sinh t}{2i(\cos t\cosh t + \sin t\sinh t)}\right) \end{aligned} \tag{3.25}$$

obeys the symmetry under the substitution $x \leftrightarrow y$. The initial condition is the standing wave $K(x,y,0) = e^{ixy}/\sqrt{2\pi}$.

In a similar fashion, the characteristic equations (2.20) and (2.14) have a common solution

$$\mu = -\mu_3 = \cos t\cosh t - \sin t\sinh t. \tag{3.26}$$

The coefficients of the corresponding Hamiltonians are

$$a_1 = \sin^2 t, \quad b_1 = \cos^2 t, \quad c_1 = 2d_1 = -\sin(2t) \tag{3.27}$$

and

$$a_2 = \cosh^2 t, \quad b_2 = \sinh^2 t, \quad c_2 = 2d_2 = -\sinh(2t) . \tag{3.28}$$

The criteria (3.4) are satisfied once again and the particular solution is given by

$$\psi = K(x,y,t) = \frac{1}{\sqrt{2\pi(\cos t\cosh t - \sin t\sinh t)}} \tag{3.29}$$

$$\times \exp\left(\frac{\left(x^2+y^2\right)\sin t\cosh t+2xy+\left(x^2-y^2\right)\cos t\sinh t}{2i\left(\cos t\cosh t-\sin t\sinh t\right)}\right)$$

with $K(x,y,0)=e^{-ixy}/\sqrt{2\pi}$. We shall discuss in section 6 how these solutions are related to the corresponding time evolution operators.

4. Complex Form of The Propagators

It is worth noting that the propagator in (1.5) can be rewritten in terms of the Wronskians of trigonometric and hypergeometric functions as

$$\begin{aligned} G(x,y,t) &= \frac{1}{\sqrt{2\pi i W(\cos t,\cosh t)}} \\ &\quad\times\exp\left(\frac{W(\sin t,\cosh t)\,x^2+2xy-W(\cos t,\sinh t)\,y^2}{2iW(\cos t,\cosh t)}\right). \end{aligned} \tag{4.1}$$

This simply means that our propagator has the following structure

$$G=\sqrt{\frac{c_2-c_3}{4\pi i}}\;\exp\left(\frac{c_1x^2+(c_2-c_3)\,xy-c_4y^2}{2i}\right), \tag{4.2}$$

where the coefficients are solutions of the system of linear equations

$$\begin{aligned} c_1\cos t+c_2\cosh t &= \sin t, \\ -c_1\sin t+c_2\sinh t &= \cos t, \\ c_3\cos t+c_4\cosh t &= \sinh t, \\ -c_3\sin t+c_4\sinh t &= \cosh t \end{aligned} \tag{4.3}$$

obtained by Cramer's rule. A complex form of this system is

$$c_1z^*+c_2\zeta=iz^*,\quad c_3z^*+c_4\zeta=i\zeta^*, \tag{4.4}$$

where we introduce two complex variables

$$z=\cos t+i\sin t,\qquad \zeta=\cosh t+i\sinh t \tag{4.5}$$

and use the star for complex conjugate. Taking the complex conjugate of the system (4.4), which has the real-valued solutions, and using Cramer's rule once again, one gets

$$\begin{aligned} c_1&=\frac{z\zeta+z^*\zeta^*}{i\left(z\zeta-z^*\zeta^*\right)}, &\qquad c_2&=\frac{2i}{z\zeta-z^*\zeta^*}, \\ c_3&=\frac{2}{i\left(z\zeta-z^*\zeta^*\right)}, &\qquad c_4&=-\frac{z\zeta^*+z^*\zeta}{i\left(z\zeta-z^*\zeta^*\right)}. \end{aligned} \tag{4.6}$$

As a result, we obtain a compact symmetric expression of the propagator (1.5) as a function of two complex variables

$$\begin{aligned} G(x,y,t) &= G(x,y,z,\zeta)=\frac{1}{\sqrt{\pi\left(z\zeta-z^*\zeta^*\right)}} \\ &\quad\times\exp\left(\frac{\left(z\zeta+z^*\zeta^*\right)x^2-4xy+\left(z\zeta^*+z^*\zeta\right)y^2}{2\left(z^*\zeta^*-z\zeta\right)}\right). \end{aligned} \tag{4.7}$$

This function takes a familiar form

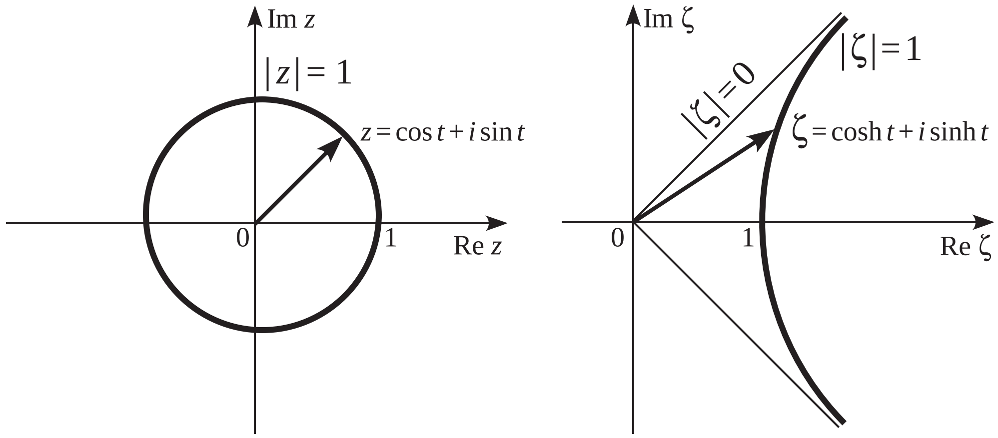

FIGURE 1. Two synchronized "clocks", or contours in complex Euclidean z and pseudo-Euclidean ζ "time" planes, corresponding to modified oscillators.

$$G=\frac{1}{\sqrt{2\pi i\left(x_1x_4+x_2x_3\right)}}\exp\left(\frac{\left(x^2-y^2\right)x_2x_4+2xy-\left(x^2+y^2\right)x_1x_3}{2i\left(x_1x_4+x_2x_3\right)}\right), \tag{4.8}$$

in a real-valued four-dimensional vector space, if we set $z=x_1+ix_2$ and $\zeta=x_3+ix_4$ with $x_1'=-x_2$, $x_2'=x_1$, $x_3'=x_4$, $x_4'=x_3$, and solve the following initial value problem

$$\begin{aligned} &x_1''+x_1=0, \qquad x_1(0)=1, \quad x_1'(0)=0,\\ &x_2''+x_2=0, \qquad x_2(0)=0, \quad x_2'(0)=1,\\ &x_3''-x_3=0, \qquad x_3(0)=1, \quad x_3'(0)=0,\\ &x_4''-x_4=0, \qquad x_4(0)=0, \quad x_4'(0)=1, \end{aligned} \tag{4.9}$$

the solution of which can be interpreted as a trajectory of a classical "particle" moving in this space; cf. (1.5).

It is worth noting that our propagators expression (4.7), extended to a function of two independent complex variables z and ζ, allows us to unify several exactly solvable quantum mechanical models in geometrical terms, namely, by choosing different contours, with certain "synchronized" parametrization, in the pair of complex "time" planes under consideration. Indeed, the free particle propagator (3.12) appears in this way when one chooses $z=1$ and $\zeta=1+it$. The simple harmonic oscillator propagator (3.13) corresponds to the case $z=1$ and $\zeta=e^{it}$. As we have seen in this section, the propagator (1.5) is also a special case of (4.7). This is why we may refer to the Hamiltonians under consideration as the ones of modified oscillators. By a vague analogy with the special theory of relativity, one may also say that in this case there are two synchronized "clocks", namely, the two contour parameterized by (4.5), one in Euclidean and another one in the pseudo-Euclidean two dimensional spaces, respectively, which geometrically describes a time evolution for the Hamiltonians of modified oscillators; see Figure 1. This idea of introducing a geometric structure of time in the problem under consideration may be useful for other types of evolutionary equations.

In a similar fashion, our new propagator (2.23) can be rewritten in terms of the Wronskians as

$$G(x,y,t) \quad = \quad \frac{1}{\sqrt{2\pi iW(\sin t,\sinh t)}} \tag{4.10}$$

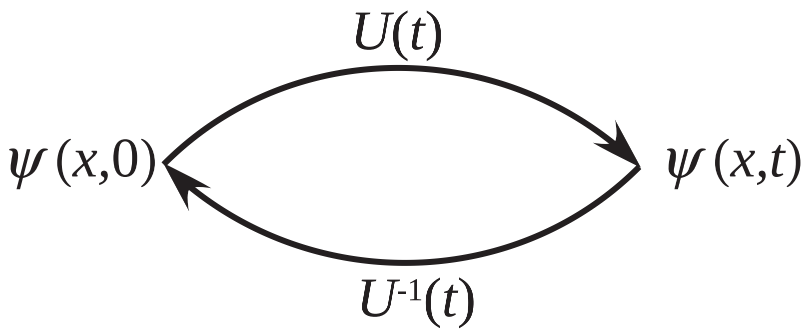

FIGURE 2. The time evolution operator and its inverse.

$$\times \exp\left(\frac{W\left(\cos t,\sinh t\right)x^2-2xy-W\left(\sin t,\cosh t\right)y^2}{2iW\left(\sinh t,\sin t\right)}\right).$$

The corresponding complex form is

$$\begin{aligned} G\left(x,y,t\right) &= G\left(x,y,z,\zeta\right)=\frac{1}{\sqrt{\pi\left(z\zeta^*-z^*\zeta\right)}} \\ &\quad \times \exp\left(\frac{\left(z\zeta^*+z^*\zeta\right)x^2-4xy+\left(z\zeta+z^*\zeta^*\right)y^2}{2\left(z^*\zeta-z\zeta^*\right)}\right). \end{aligned} \tag{4.11}$$

We leave the details to the reader.

5. The Inverse Operator and Time Reversal

We follow the method suggested in [61] for general quadratic Hamiltonians with somewhat different details. The left inverse of the integral operator defined by (1.4)–(1.5), namely,

$$U\left(t\right)\psi\left(x\right)=\int_{-\infty}^{\infty}G\left(x,y,t\right)\ \psi\left(y\right)\ dy, \tag{5.1}$$

is

$$U^{-1}\left(t\right)\chi\left(x\right)=\int_{-\infty}^{\infty}G\left(y,x,-t\right)\ \chi\left(y\right)\ dy \tag{5.2}$$

in view of $U^{-1}=U^{\dagger}$; see Figure 2. Indeed, when $s<t$, by the Fubini theorem

$$\begin{aligned} U^{-1}\left(s\right)\left(U\left(t\right)\psi\right) &= U^{-1}\left(s\right)\chi=\int_{-\infty}^{\infty}G\left(z,x,-s\right)\ \chi\left(z\right)\ dz \\ &= \int_{-\infty}^{\infty}G\left(z,x,-s\right)\ \left(\int_{-\infty}^{\infty}G\left(z,y,t\right)\ \psi\left(y\right)\ dy\right)\ dz \\ &= \int_{-\infty}^{\infty}G\left(x,y,s,t\right)\ \psi\left(y\right)\ dy. \end{aligned} \tag{5.3}$$

Here

$$G\left(x,y,s,t\right) = \int_{-\infty}^{\infty}G\left(z,x,-s\right)G\left(z,y,t\right)\ dz \tag{5.4}$$

$$\begin{aligned} &= \frac{e^{i\left(\gamma(t)y^2-\gamma(s)x^2\right)}}{2\pi\sqrt{\mu(s)\mu(t)}} \int_{-\infty}^{\infty} e^{i\left((\alpha(t)-\alpha(s))z^2+(\beta(t)y-\beta(s)x)z\right)}\ dz \\ &= \frac{1}{\sqrt{4\pi i\mu(s)\mu(t)(\alpha(s)-\alpha(t))}} \\ &\quad \times \exp\left(\frac{(\beta(t)y-\beta(s)x)^2-4(\alpha(t)-\alpha(s))(\gamma(t)y^2-\gamma(s)x^2)}{4i(\alpha(t)-\alpha(s))}\right) \end{aligned}$$

by the familiar Gaussian integral [6], [48] and [56]:

$$\int_{-\infty}^{\infty} e^{i\left(az^2+2bz\right)}\ dz = \sqrt{\frac{\pi i}{a}}\, e^{-ib^2/a}. \tag{5.5}$$

In view of (2.6),

$$(\beta(t)y-\beta(s)x)^2 = \left(\frac{x}{\mu(s)}-\frac{y}{\mu(t)}\right)^2 = \frac{\left(\sqrt{\frac{\mu(t)}{\mu(s)}}x-\sqrt{\frac{\mu(s)}{\mu(t)}}y\right)^2}{\mu(s)\mu(t)(\alpha(t)-\alpha(s))}, \tag{5.6}$$

and a singular part of (5.4) becomes

$$\begin{aligned} &\frac{1}{\sqrt{4\pi i\mu(s)\mu(t)(\alpha(s)-\alpha(t))}} \exp\left(\frac{(\beta(t)y-\beta(s)x)^2}{4i(\alpha(t)-\alpha(s))}\right) \\ &\quad = \frac{1}{\sqrt{4\pi i\mu(s)\mu(t)(\alpha(s)-\alpha(t))}} \exp\left(\frac{\left(\sqrt{\frac{\mu(t)}{\mu(s)}}x-\sqrt{\frac{\mu(s)}{\mu(t)}}y\right)^2}{4i\mu(s)\mu(t)(\alpha(t)-\alpha(s))}\right). \end{aligned}$$

Thus, in the limit $s\rightarrow t^-$, one can obtain formally the identity operator in the right hand side of (5.3). The leave the details to the reader.

On the other hand, the integral operator in (1.4)–(1.5), namely,

$$\chi(x) = \frac{1}{\sqrt{2\pi i\mu(t)}} \int_{-\infty}^{\infty} e^{i\left(\alpha(t)x^2+\beta(t)xy+\gamma(t)y^2\right)}\ \psi(y)\ dy \tag{5.7}$$

is essentially the Fourier transform and its inverse is given by

$$\psi(y) = \frac{1}{\sqrt{-2\pi i\mu(t)}} \int_{-\infty}^{\infty} e^{-i\left(\alpha(t)x^2+\beta(t)xy+\gamma(t)y^2\right)}\ \chi(x)\ dy \tag{5.8}$$

in correspondence with our definition (5.2) in view of (2.6).

The Schrödinger equation (1.1) retains the same form if we replace t in it by $-t$ and, at the same time, take complex conjugate provided that $(H(-t)\varphi)^* = H(t)\varphi^*$. The last relation holds for both Hamiltonians (1.2) and (1.7). Hence the function $\chi(x,t) = \psi^*(x,-t)$ does satisfy the same equation as the original wave function $\psi(x,t)$. This property is usually known as the symmetry with respect to time inversion (time reversal) in quantum mechanics [29], [36], [42], [69]. This fact is obvious from a general solution given by (1.4)–(1.5) for our Hamiltonians.

On the other hand, by the definition, the (left) inverse $U^{-1}(t)$ of the time evolution operator $U(t)$ returns the system to its initial quantum state:

$$\psi(x,t) = U(t)\,\psi(x,0)\,, \tag{5.9}$$

$$U^{-1}(t)\,\psi(x,t) = U^{-1}(t)\,(U(t)\,\psi(x,0)) = \psi(x,0)\,. \tag{5.10}$$

Our analysis of two oscillator models under consideration shows that this may be related to the reversal of time in the following manner. The left inverse of the time evolution operator (1.4) for the Schrödinger equation (1.1) with the original Hamiltonian of a modified oscillator (1.2) can be obtained by the time inversion $t \to -t$ in the evolution operator corresponding to the new "dual" Hamiltonian (1.7) (and vise versa). The same is true for the second pair of dual Hamiltonians. More details will be given in section 6. This is an example of a situation in mathematical physics and quantum mechanics when a change in the direction of time may require a total change of the system dynamics in order to return the system back to its original quantum state. Moreover, moving backward in time the system will repeat the same quantum states only when

$$\psi(x,t-s) = U(t-s)\,\psi(x,0) = U^{-1}(s)\,U(t)\,\psi(x,0)\,, \qquad 0 \le s \le t, \tag{5.11}$$

which is equivalent to the semi-group property

$$U(s)\,U(t-s) = U(t) \tag{5.12}$$

for the time evolution operator. This seems not true for propagators (1.5) and (2.23).

6. The Momentum Representation

The time-dependent Schrödinger equation (2.1) can be rewritten in terms of the operator of coordinate x and the operator of linear momentum $p_x = i^{-1}\partial/\partial x$ as follows

$$i\frac{\partial\psi}{\partial t} = \big(a(t)\,p_x^2 + b(t)\,x^2 + d(t)\,(xp_x + p_x x)\big)\,\psi \tag{6.1}$$

with $c = 2d$. The corresponding quadratic Hamiltonian

$$H = a(t)\,p_x^2 + b(t)\,x^2 + d(t)\,(xp_x + p_x x) \tag{6.2}$$

obeys a special symmetry, namely, it formally preserves this structure under the permutation $x \leftrightarrow p_x$. This fact is well-known for the simple harmonic oscillator [29], [36], [42].

In order to interchange the coordinate and momentum operators in quantum mechanics one switches between the coordinate and momentum representations by means of the Fourier transform

$$\psi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ixy}\chi(y)\ dy = F[\chi] \tag{6.3}$$

and its inverse

$$\chi(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ixy}\psi(x)\ dx = F^{-1}[\psi]\,. \tag{6.4}$$

Indeed, the familiar properties

$$p_x\psi = p_x F[\chi] = F[y\chi]\,, \qquad x\psi = xF[\chi] = -F[p_y\chi] \tag{6.5}$$

imply

$$p_x^2\psi = F\left[y^2\chi\right], \qquad x^2\psi = F\left[p_y^2\chi\right] \tag{6.6}$$

and

$$\left(xp_x + p_x x\right)\psi = -F\left[\left(p_y y + y p_y\right)\chi\right]. \tag{6.7}$$

Therefore,

$$\begin{aligned} H\psi &= \left(ap_x^2 + bx^2 + d\left(xp_x + p_x x\right)\right) F\left[\chi\right] \\ &= F\left[\left(bp_y^2 + ay^2 - d\left(yp_y + p_y y\right)\right)\chi\right] \end{aligned}$$

by the linearity of the Fourier transform. In view of

$$\frac{\partial\psi}{\partial t} = F\left[\frac{\partial\chi}{\partial t}\right]$$

the Schrödinger equation (6.1) takes the form

$$i\frac{\partial\chi}{\partial t} = \left(b\left(t\right)p_y^2 + a\left(t\right)y^2 - d\left(t\right)\left(yp_y + p_y y\right)\right)\chi \tag{6.8}$$

with $a \leftrightarrow b$ and $d \rightarrow -d$ in the momentum representation.

This property finally reveals that our quadratic Hamiltonians (2.2) and (2.18), similarly (2.3) and (2.19), corresponds to the same Schrödinger equation written in the coordinate and momentum representations, respectively. Thus, in section 2, we have solved the Cauchy initial value problem for modified oscillators both in the coordinate and momentum representations.

In this paper the creation and annihilation operators are defined by

$$a^{\dagger} = \frac{p_x + ix}{\sqrt{2}} = \frac{1}{i\sqrt{2}}\left(\frac{\partial}{\partial x} - x\right), \tag{6.9}$$

$$a = \frac{p_x - ix}{\sqrt{2}} = \frac{1}{i\sqrt{2}}\left(\frac{\partial}{\partial x} + x\right) \tag{6.10}$$

with the familiar commutator $\left[a, a^{\dagger}\right] = aa^{\dagger} - a^{\dagger}a = 1$ [28]. One can see that

$$\begin{aligned} a_x\psi &= a_x F\left[\chi\right] = F\left[ia_y\chi\right], \\ a_x^{\dagger}\psi &= a_x^{\dagger}F\left[\chi\right] = F\left[-ia_y^{\dagger}\chi\right], \end{aligned} \tag{6.11}$$

or

$$a_x \rightarrow ia_y, \qquad a_x^{\dagger} \rightarrow -ia_y^{\dagger} \tag{6.12}$$

under the Fourier transform. This observation will be important in the next section.

Finally one can summarize all results on solution of the Cauchy initial value problems for the modified oscillator under consideration in a form of the commutative evolution diagram on Figure 3. We denote

$$U\left(t\right)\psi\left(x\right) = \int_{-\infty}^{\infty} G_U\left(x,y,t\right)\ \psi\left(y\right)\ dy, \tag{6.13}$$

$$K\left(t\right)\psi\left(x\right) = \int_{-\infty}^{\infty} K_U\left(x,y,t\right)\ \psi\left(y\right)\ dy, \tag{6.14}$$

$$V\left(t\right)\psi\left(x\right) = \int_{-\infty}^{\infty} G_V\left(x,y,t\right)\ \psi\left(y\right)\ dy, \tag{6.15}$$

$$L\left(t\right)\psi\left(x\right) = \int_{-\infty}^{\infty} K_V\left(x,y,t\right)\ \psi\left(y\right)\ dy. \tag{6.16}$$

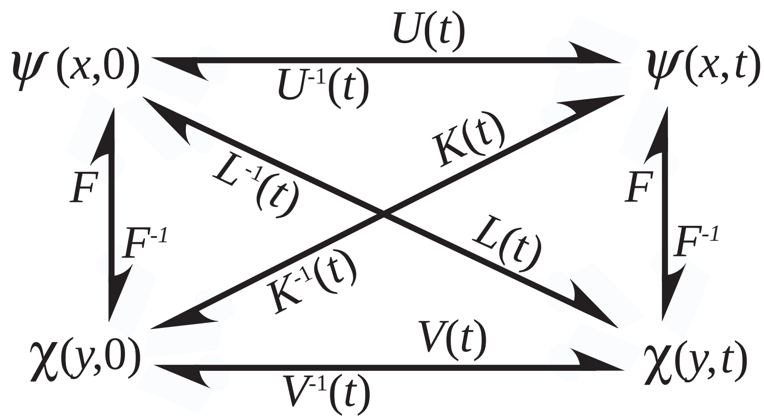

FIGURE 3. The commutative evolution diagram in $\boldsymbol{R}$.

The kernels of these integral operators are defined as follows. Here $G_U(x,y,t)$ and $G_V(x,y,t)$ are the Green functions in (1.5) and (2.23), respectively. The kernels $K_U(x,y,t)$ and $K_V(x,y,t)$ are given by (3.25) and (3.29), respectively. The following operator identities hold

$$U(t) = K(t)F^{-1} = FL(t) = FV(t)F^{-1}, \tag{6.17}$$
$$V(t) = L(t)F = F^{-1}K(t) = F^{-1}U(t)F, \tag{6.18}$$
$$U^{-1}(t) = FK^{-1}(t) = L^{-1}(t)F^{-1} = FV^{-1}(t)F^{-1}, \tag{6.19}$$
$$V^{-1}(t) = F^{-1}L^{-1}(t) = K^{-1}(t)F = F^{-1}U^{-1}(t)F, \tag{6.20}$$
$$K(t) = FL(t)F, \qquad L(t) = F^{-1}K(t)F^{-1}, \tag{6.21}$$
$$K^{-1}(t) = F^{-1}L^{-1}(t)F^{-1}, \quad L^{-1}(t) = FK^{-1}(t)F. \tag{6.22}$$

Here F and F^{-1} are the operators of Fourier transform and its inverse, respectively, which relate the wave functions in the coordinate and momentum representations

$$\psi = F[\chi], \qquad \chi = F^{-1}[\psi]$$

at any given moment of time thus representing the vertical arrows at our diagram. The time evolution operators $U(t)$, $V(t)$ and their inverses $U^{-1}(t)$, $V^{-1}(t)$ correspond to the horizontal arrows in the coordinate and momentum representations, respectively. They obey the symmetry with respect to the time reversal, which has been discussed in section 5.

In order to discuss the diagonal arrows of the time evolution diagram on Figure 3, we have to go back, say, to the particular solution (3.25). A more general solution of the Schrödinger equation (2.1)–(2.2) can be obtained by the superposition principle in the form

$$\psi(x,t) = \int_{-\infty}^{\infty} K_U(x,y,t)\ \chi(y,0)\ dy, \tag{6.23}$$

where χ is an arbitrary function, independent of time, such that the integral converges and one can interchange the differentiation and integration. In view of the continuity of the kernel at $t=0$, we get

$$\psi(x,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixy}\ \chi(y,0)\ dy, \tag{6.24}$$

which simply relates the initial data in the coordinate and momentum representations. Then solution of the initial value problem is given by the inverse of the Fourier transform

$$\chi(y,0)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-ixy}\ \psi(x,0)\ dx \tag{6.25}$$

followed by the back substitution of this expression into (6.23). This implies the above factorization $U(t)=K(t)F^{-1}$ of the corresponding time evolution operator; see (6.17). The Green function (1.5) can be derived as

$$G_U(x,y,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}K_U(x,z,t)\ e^{-iyz}\ dz \tag{6.26}$$

with the help of the integral (5.5). The second equation, $U(t)=FL(t)$, is related to following integral

$$G_U(x,y,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ixz}\ K_V(z,y,t)\ dz. \tag{6.27}$$

The meaning of the operator $L(t)$, represented by another diagonal arrow on the time evolution diagram, is established in a similar fashion. One can see that the relation $V(t)=L(t)F$ in (6.18) follows from the elementary integral

$$G_V(x,y,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}K_V(x,z,t)\ e^{iyz}\ dz \tag{6.28}$$

and $K(t)=FV(t)$ corresponds to

$$K_U(x,y,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ixz}\ G_V(z,y,t)\ dz. \tag{6.29}$$

This proves (6.17)–(6.18). The inverses $K^{-1}(t)$ and $L^{-1}(t)$ are found, for instance, by the inverse of Fourier transform similar to (5.7)–(5.8). They are not directly related to the reversal of time. We leave further details about the structure of the commutative diagram on Figure 3 to the reader.

7. The Case of n-Dimensions

In the case of $\boldsymbol{R}^n$ with an arbitrary number of dimensions, the Schrödinger equation for a modified oscillator (1.1) with the original Hamiltonian

$$H(t)=\frac{1}{2}\sum_{s=1}^{n}\left(a_s a_s^{\dagger}+a_s^{\dagger}a_s\right)+\frac{1}{2}e^{2it}\sum_{s=1}^{n}\left(a_s\right)^2+\frac{1}{2}e^{-2it}\sum_{s=1}^{n}\left(a_s^{\dagger}\right)^2, \tag{7.1}$$

considered by Meiler, Cordero-Soto, and Suslov [40], has the Green function of the form

$$\begin{aligned} G_t(\boldsymbol{x},\boldsymbol{x}') &= \prod_{s=1}^{n}G_t(x_s,x_s') \qquad (7.2)\\ &= \left(\frac{1}{2\pi i(\cos t\sinh t+\sin t\cosh t)}\right)^{n/2}\\ &\quad\times\exp\left(\frac{(\boldsymbol{x}^2-\boldsymbol{x}'^2)\sin t\sinh t+2\boldsymbol{x}\cdot\boldsymbol{x}'-(\boldsymbol{x}^2+\boldsymbol{x}'^2)\cos t\cosh t}{2i(\cos t\sinh t+\sin t\cosh t)}\right). \end{aligned}$$

Solution of the Cauchy initial value problem can be written as

$$\psi(\boldsymbol{x},t)=\int_{\boldsymbol{R}^n} G_t(\boldsymbol{x},\boldsymbol{x}')\ \psi(\boldsymbol{x}',0)\ d\boldsymbol{x}', \tag{7.3}$$

where $dv'=d\boldsymbol{x}'=dx_1'\cdot\ldots\cdot dx_n'$. The propagator expansion in the hyperspherical harmonics is given by

$$G_t(\boldsymbol{x},\boldsymbol{x}')=\sum_{K\nu}Y_{K\nu}(\Omega)\ Y^*_{K\nu}(\Omega')\ \mathcal{G}^K_t(r,r') \tag{7.4}$$

with

$$\begin{aligned}\mathcal{G}^K_t(r,r') &= \frac{e^{-i\pi(2K+n)/4}}{2^{K+n/2-1}\Gamma(K+n/2)}\frac{(rr')^K}{(\cos t\sinh t+\sin t\cosh t)^{K+n/2}} \\ &\quad\times\exp\left(i\frac{\left(r^2+(r')^2\right)\cos t\cosh t-\left(r^2-(r')^2\right)\sin t\sinh t}{2(\cos t\sinh t+\sin t\cosh t)}\right) \\ &\quad\times\ {}_0F_1\left(\begin{matrix}-\\ K+n/2\end{matrix};\ -\frac{(rr')^2}{4(\cos t\sinh t+\sin t\cosh t)^2}\right).\end{aligned} \tag{7.5}$$

Here $Y_{K\nu}(\Omega)$ are the hyperspherical harmonics constructed by the given tree T in the graphical approach of Vilenkin, Kuznetsov and Smorodinskiĭ [46], the integer K corresponds to the constant of separation of the variables at the root of T (denoted by K due to the tradition of the method of K-harmonics in nuclear physics [59]) and $\nu=\{l_1,l_2,\ldots,l_p\}$ is the set of all other subscripts corresponding to the remaining vertexes of the binary tree T. These formulas imply the familiar expansion of a plane wave in $\boldsymbol{R}^n$ in terms of the hyperspherical harmonics

$$e^{i\boldsymbol{x}\cdot\boldsymbol{x}'}=rr'\left(\frac{2\pi}{rr'}\right)^{n/2}\sum_{K\nu}i^K\ Y^*_{K\nu}(\Omega)\ Y_{K\nu}(\Omega')\ J_{K+n/2-1}(rr'), \tag{7.6}$$

where

$$J_\mu(z)=\frac{(z/2)^\mu}{\Gamma(\mu+1)}\ {}_0F_1\left(\begin{matrix}-\\ \mu+1\end{matrix};\ -\frac{z^2}{4}\right) \tag{7.7}$$

is the Bessel function. See [40] and references therein for more details. It is worth noting that the Green function (1.5) was originally found by Meiler, Cordero-Soto, and Suslov as the special case $n=1$ of the expansion (7.4)–(7.5). The dynamical $SU(1,1)$ symmetry of the harmonic oscillator wave functions, Bargmann's functions for the discrete positive series of the irreducible representations of this group, the Fourier integral of a weighted product of the Meixner–Pollaczek polynomials, a Hankel-type integral transform and the hyperspherical harmonics were utilized in order to derive the n-dimensional Green function.

Our results show that the "dual" Schrödinger equation (1.6) with a new Hamiltonian of the form

$$H(\tau)=\frac{1}{2}\sum_{s=1}^n\left(a_sa_s^\dagger+a_s^\dagger a_s\right)+\frac{1}{2}e^{-i\arctan(2\tau)}\sum_{s=1}^n(a_s)^2+\frac{1}{2}e^{i\arctan(2\tau)}\sum_{s=1}^n\left(a_s^\dagger\right)^2 \tag{7.8}$$

has the propagator that is almost identical to (7.2) but with $\boldsymbol{x}\leftrightarrow\boldsymbol{x}'$. Indeed, in the case of n-dimensions one has

$$H(\tau)=\sum_{s=1}^n H_s(\tau), \tag{7.9}$$

where we denote

$$H_s(\tau) = \frac{1}{2}\left(a_s a_s^\dagger + a_s^\dagger a_s\right) + \frac{1}{2}e^{-i\arctan(2\tau)}\left(a_s\right)^2 + \frac{1}{2}e^{i\arctan(2\tau)}\left(a_s^\dagger\right)^2. \tag{7.10}$$

If one chooses

$$\begin{aligned} \psi_s &= \psi_s(x_s, t) = G_t(x_s', x_s) \\ &= \left(\frac{1}{2\pi i(\cos t\sinh t + \sin t\cosh t)}\right)^{1/2} \\ &\quad\times\exp\left(\frac{(x_s'^2 - x_s^2)\sin t\sinh t + 2x_s' x_s - (x_s'^2 + x_s^2)\cos t\cosh t}{2i(\cos t\sinh t + \sin t\cosh t)}\right) \end{aligned} \tag{7.11}$$

with

$$i\frac{\partial\psi_s}{\partial\tau} = H_s(\tau)\,\psi_s, \qquad t = \frac{1}{2}\sinh(2\tau) \tag{7.12}$$

and denotes

$$\psi = \prod_{k=1}^n \psi_k = \prod_{k=1}^n G_t(x_k', x_k) = G_t(\boldsymbol{x}', \boldsymbol{x}), \tag{7.13}$$

then

$$i\frac{\partial\psi}{\partial\tau} = \sum_{s=1}^n \left(i\frac{\partial\psi_s}{\partial\tau}\right)\prod_{k\neq s}\psi_k \tag{7.14}$$

and

$$H(\tau)\,\psi = \sum_{s=1}^n \left(H_s(\tau)\,\psi_s\right)\prod_{k\neq s}\psi_k. \tag{7.15}$$

As a result,

$$\left(i\frac{\partial}{\partial\tau} - H(\tau)\right)\psi = \sum_{s=1}^n\left(i\frac{\partial\psi_s}{\partial\tau} - H_s(\tau)\,\psi_s\right)\prod_{k\neq s}\psi_k \equiv 0, \tag{7.16}$$

and Eq. (1.6) for the n-dimensional propagator is satisfied. For the initial data, formally,

$$\lim_{t\to0^+} G_t(\boldsymbol{x}', \boldsymbol{x}) = \prod_{k=1}^n \lim_{t\to0^+} G_t(x_k', x_k) = \prod_{k=1}^n \delta(x_k' - x_k) = \delta(\boldsymbol{x} - \boldsymbol{x}'), \tag{7.17}$$

where $\delta(\boldsymbol{x})$ is the Dirac delta function in $\boldsymbol{R}^n$. Further details are left to the reader.

The n-dimensional version of the Hamiltonian corresponding to the coefficients (2.18) is given by

$$H(t) = \frac{1}{2}\sum_{s=1}^n\left(a_s a_s^\dagger + a_s^\dagger a_s\right) - \frac{1}{2}e^{2it}\sum_{s=1}^n\left(a_s\right)^2 - \frac{1}{2}e^{-2it}\sum_{s=1}^n\left(a_s^\dagger\right)^2 \tag{7.18}$$

with the propagator

$$\begin{aligned} G_t(\boldsymbol{x}, \boldsymbol{x}') &= \left(\frac{1}{2\pi i(\sin t\cosh t - \cos t\sinh t)}\right)^{n/2} \\ &\quad\times\exp\left(\frac{(\boldsymbol{x}^2 + \boldsymbol{x}'^2)\cos t\cosh t - 2\boldsymbol{x}\cdot\boldsymbol{x}' + (\boldsymbol{x}^2 - \boldsymbol{x}'^2)\sin t\sinh t}{2i(\cos t\sinh t - \sin t\cosh t)}\right). \end{aligned} \tag{7.19}$$

The dual counterpart of this Hamiltonian with respect to time reversal has the form

$$H(\tau)=\frac{1}{2}\sum_{s=1}^{n}\left(a_sa_s^{\dagger}+a_s^{\dagger}a_s\right)-\frac{1}{2}e^{-i\arctan(2\tau)}\sum_{s=1}^{n}\left(a_s\right)^2-\frac{1}{2}e^{i\arctan(2\tau)}\sum_{s=1}^{n}\left(a_s^{\dagger}\right)^2 \tag{7.20}$$

and one has to interchange $\boldsymbol{x}\leftrightarrow\boldsymbol{x}'$ in (7.19) in order to obtain the corresponding Green function. It is worth noting that the Hamiltonians (7.1) and (7.18) (respectively, (7.8) and (7.20)) are transforming into each other under the substitution $a_s\rightarrow ia_s$, $a_s^{\dagger}\rightarrow -ia_s^{\dagger}$, which preserves the commutation relations of the creation and annihilation operators. As we have seen in the previous section this property is related to solving the problem in the coordinate and momentum representations.

Combining all four cases together, one may summarize that two Hamiltonians,

$$H_{\pm}(t)=\frac{1}{2}\sum_{s=1}^{n}\left(a_sa_s^{\dagger}+a_s^{\dagger}a_s\right)\pm\frac{1}{2}e^{2it}\sum_{s=1}^{n}\left(a_s\right)^2\pm\frac{1}{2}e^{-2it}\sum_{s=1}^{n}\left(a_s^{\dagger}\right)^2, \tag{7.21}$$

and their duals with respect to the time reversal,

$$H_{\pm}(\tau)=\frac{1}{2}\sum_{s=1}^{n}\left(a_sa_s^{\dagger}+a_s^{\dagger}a_s\right)\pm\frac{1}{2}e^{-i\arctan(2\tau)}\sum_{s=1}^{n}\left(a_s\right)^2\pm\frac{1}{2}e^{i\arctan(2\tau)}\sum_{s=1}^{n}\left(a_s^{\dagger}\right)^2 \tag{7.22}$$

with $\tau=\frac{1}{2}\sinh(2t)$, have the following Green functions:

$$\begin{aligned} G_t^{\pm}(\boldsymbol{x},\boldsymbol{x}') &= \left(\frac{1}{2\pi i\left(\sin t\cosh t\pm\cos t\sinh t\right)}\right)^{n/2} \\ &\quad\times\exp\left(\frac{\pm\left(\boldsymbol{x}^2-\boldsymbol{x}'^2\right)\sin t\sinh t+2\boldsymbol{x}\cdot\boldsymbol{x}'-\left(\boldsymbol{x}^2+\boldsymbol{x}'^2\right)\cos t\cosh t}{2i\left(\sin t\cosh t\pm\cos t\sinh t\right)}\right). \end{aligned} \tag{7.23}$$

This expression is valid for two Hamiltonians (7.21), respectively. One has to interchange $\boldsymbol{x}\leftrightarrow\boldsymbol{x}'$ for the case of the dual Hamiltonians (7.22).

In a similar fashion, the n-dimensional form of the kernels (3.25) and (3.29) is

$$\begin{aligned} K_t^{\pm}(\boldsymbol{x},\boldsymbol{x}') &= \left(\frac{1}{2\pi\left(\cos t\cosh t\pm\sin t\sinh t\right)}\right)^{n/2} \\ &\quad\times\exp\left(\frac{\left(\boldsymbol{x}^2+\boldsymbol{x}'^2\right)\sin t\cosh t\mp 2\boldsymbol{x}\cdot\boldsymbol{x}'\mp\left(\boldsymbol{x}^2-\boldsymbol{x}'^2\right)\cos t\sinh t}{2i\left(\cos t\cosh t\pm\sin t\sinh t\right)}\right) \end{aligned} \tag{7.24}$$

and

$$\begin{aligned} G_t^{\pm}(\boldsymbol{x},\boldsymbol{x}') &= \frac{1}{(2\pi)^{n/2}}\int_{\boldsymbol{R}^n}K_t^{\pm}(\boldsymbol{x},\boldsymbol{x}'')\ e^{\mp i\boldsymbol{x}'\cdot\boldsymbol{x}''}\ d\boldsymbol{x}'' \\ &= \frac{1}{(2\pi)^{n/2}}\int_{\boldsymbol{R}^n}e^{\pm i\boldsymbol{x}\cdot\boldsymbol{x}''}\ K_t^{\mp}(\boldsymbol{x}'',\boldsymbol{x}')\ d\boldsymbol{x}''. \end{aligned} \tag{7.25}$$

Denoting

$$\begin{aligned} U_{\pm}(t)\,\psi(\boldsymbol{x}) &= \int_{\boldsymbol{R}^n}G_t^{\pm}(\boldsymbol{x},\boldsymbol{x}')\ \psi(\boldsymbol{x}')\ d\boldsymbol{x}', \\ K_{\pm}(t)\,\psi(\boldsymbol{x}) &= \int_{\boldsymbol{R}^n}K_t^{\pm}(\boldsymbol{x},\boldsymbol{x}')\ \psi(\boldsymbol{x}')\ d\boldsymbol{x}', \end{aligned} \tag{7.26}$$

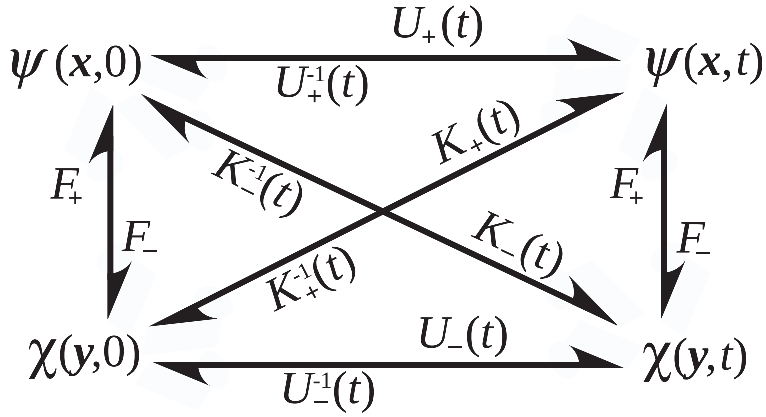

FIGURE 4. The commutative evolution diagram in $\boldsymbol{R}^n$.

and

$$F_{\pm}\psi\left(\boldsymbol{x}\right)=\frac{1}{\left(2\pi\right)^{n/2}}\int_{\boldsymbol{R}^n}e^{\pm i\boldsymbol{x}\cdot\boldsymbol{x}'}\ \psi\left(\boldsymbol{x}'\right)\ d\boldsymbol{x}' \tag{7.27}$$

one arrives at the commutative evolution diagram in $\boldsymbol{R}^n$ on Figure 4. The corresponding relations are

$$\begin{aligned} U_{\pm}\left(t\right) &= K_{\pm}\left(t\right)F_{\mp}=F_{\pm}K_{\mp}\left(t\right)=F_{\pm}U_{\mp}\left(t\right)F_{\mp}, \\ U_{\pm}^{-1}\left(t\right) &= F_{\pm}K_{\pm}^{-1}\left(t\right)=K_{\mp}^{-1}\left(t\right)F_{\mp}=F_{\pm}U_{\mp}^{-1}\left(t\right)F_{\mp}. \end{aligned} \tag{7.28}$$

We leave further details to the reader.

A certain time-dependent Schrödinger equation with variable coefficients was considered in [40] in a pure algebraic manner in connection with representations of the group $SU\left(1,1\right)$ in an abstract Hilbert space. Our Hamiltonians (7.21) and (7.22) belong to the same class thus providing new explicit realizations of this model in addition to several cases already discussed by Meiler, Cordero-Soto, and Suslov.

8. Eigenfunction Expansions

The normalized wave functions of the n-dimensional harmonic oscillator

$$H_0\Psi=E\Psi,\qquad H_0=\frac{1}{2}\sum_{s=1}^{n}\left(-\frac{\partial^2}{\partial x_s^2}+x_s^2\right) \tag{8.1}$$

have the form

$$\Psi\left(\boldsymbol{x}\right)=\Psi_{NK\nu}\left(r,\Omega\right)=Y_{K\nu}\left(\Omega\right)\ R_{NK}\left(r\right), \tag{8.2}$$

where $Y_{K\nu}\left(\Omega\right)$ are the hyperspherical harmonics associated with a binary tree T, the integer number K corresponds to the constant of separation of the variables at the root of T and $\nu=\{l_1,l_2,\dots,l_p\}$ is the set of all other subscripts corresponding to the remaining vertexes of the binary tree T; see [46], [59], [67] for a graphical approach of Vilenkin, Kuznetsov and Smorodinskiĭ to the theory of spherical harmonics. The radial functions are given by

$$R_{NK}\left(r\right)=\sqrt{\frac{2\left[\left(N-K\right)/2\right]!}{\Gamma\left[\left(N+K+n\right)/2\right]}}\ \exp\left(-r^2/2\right)\ r^K\ L_{(N-K)/2}^{K+n/2-1}\left(r^2\right), \tag{8.3}$$

where $L_k^\alpha(\xi)$ are the Laguerre polynomials. The corresponding energy levels are

$$E = E_N = N + n/2, \qquad (N-K)/2 = k = 0, 1, 2, ... \tag{8.4}$$

and we can use the $SU(1,1)$-notation for the wave function as follows

$$\psi_{jm\{\nu\}}(\boldsymbol{x}) = \Psi_{NK\nu}(r,\Omega) = Y_{K\nu}(\Omega)\ \ R_{NK}(r), \tag{8.5}$$

where the new quantum numbers are given by $j = K/2+n/4-1$ and $m = N/2+n/4$ with $m = j+1$, $j+2, ...$. The inequality $m \geq j+1$ holds because of the quantization rule (8.4), which gives $N = K$, $K+2$, $K+4, ...$. See [40], [46] and [59] for more details on the group theoretical properties of the n-dimensional harmonic oscillator wave functions.

The Cauchy initial value problem for the Schrödinger equation (1.1) with the Hamiltonian of a modified oscillator (7.1) has also the eigenfunction expansion form of the solution [40]:

$$\psi(\boldsymbol{x},t) = \sum_{j\{\nu\}} \sum_{m=j+1}^{\infty} c_m(t)\ \ \psi_{jm\{\nu\}}(\boldsymbol{x}) \tag{8.6}$$

with the time dependent coefficients

$$c_m(t) = e^{-2imt} \sum_{m'=j+1}^{\infty} i^{m'-m} v_{m'm}^j(2t) \int_{\boldsymbol{R}^n} \psi_{jm'\{\nu\}}^*(\boldsymbol{x}')\,\psi(\boldsymbol{x}',0)\ \ dv' \tag{8.7}$$

given in terms of the Bargmann functions [4], [46] and [67]

$$\begin{aligned} v_{mm'}^j(\mu) = {} & \frac{(-1)^{m-j-1}}{\Gamma(2j+2)} \sqrt{\frac{(m+j)!\,(m'+j)!}{(m-j-1)!\,(m'-j-1)!}}\ \left(\sinh\frac{\mu}{2}\right)^{-2j-2} \left(\tanh\frac{\mu}{2}\right)^{m+m'} \\ & \times\ \ {}_2F_1\left(\begin{array}{c} -m+j+1,\ -m'+j+1 \\ 2j+2 \end{array};\ -\frac{1}{\sinh^2(\mu/2)} \right). \end{aligned} \tag{8.8}$$

Choosing the initial data in (7.3) and (8.6)–(8.7) as $\psi(\boldsymbol{x},0) = \delta(\boldsymbol{x}-\boldsymbol{x}')$, we arrive at the eigenfunction expansion for the Green function

$$G_t(\boldsymbol{x},\boldsymbol{x}') = \sum_{j\{\nu\}} \sum_{m,m'=j+1}^{\infty} e^{-2imt}\ i^{m'-m}\ v_{m'm}^j(2t)\ \ \psi_{jm\{\nu\}}(\boldsymbol{x})\,\psi_{jm'\{\nu\}}^*(\boldsymbol{x}'), \tag{8.9}$$

where by (7.19) the following symmetry property holds

$$G_t(\boldsymbol{x},\boldsymbol{x}') = G_{-t}^*(\boldsymbol{x},\boldsymbol{x}'). \tag{8.10}$$

In this paper we have found solution of the Cauchy initial value problem for the new Hamiltonian (7.8) in an integral form

$$\psi(\boldsymbol{x},t) = \int_{\boldsymbol{R}^n} G_t(\boldsymbol{x}',\boldsymbol{x})\ \ \psi(\boldsymbol{x}',0)\ \ d\boldsymbol{x}'. \tag{8.11}$$

In view of (8.9)–(8.10), the eigenfunction expansion of this solution is given by

$$\psi(\boldsymbol{x},t) = \sum_{j\{\nu\}} \sum_{m=j+1}^{\infty} c_m(t)\ \ \psi_{jm\{\nu\}}(\boldsymbol{x}), \tag{8.12}$$

where

$$c_m(t) = \sum_{m'=j+1}^{\infty} (-i)^{m-m'} e^{-2im't} \left(v^j_{m'm}(-2t)\right)^* \int_{\boldsymbol{R}^n} \psi^*_{jm'\{\nu\}}(\boldsymbol{x}')\, \psi(\boldsymbol{x}', 0)\ dv'. \tag{8.13}$$

This expansion is in agreement with the unitary infinite matrix of the inverse operator in the basis of the harmonic oscillator wave functions; see section 5.

The cases of the Hamiltonian (7.18) and its dual (7.20) can be investigated by taking the Fourier transform of the expansions (8.6)–(8.7) and (8.12)–(8.13), respectively. The corresponding transformations of the oscillator wave functions are

$$i^{\pm N}\Psi_{NK\nu}(\boldsymbol{x}) = \frac{1}{(2\pi)^{n/2}} \int_{\boldsymbol{R}^n} e^{\pm i\boldsymbol{x}\cdot\boldsymbol{x}'}\ \Psi_{NK\nu}(\boldsymbol{x}')\ d\boldsymbol{x}'. \tag{8.14}$$

This can be evaluated in hyperspherical coordinates with the help of expansion (7.6)–(7.7), or by adding the $SU(1,1)$-momenta according to the tree T [46] and using linearity of the Fourier transform. One can use

$$e^{i\boldsymbol{x}\cdot\boldsymbol{x}'} = (2\pi)^{n/2} \sum_{K\nu} i^K\ Y^*_{K\nu}(\Omega)\ Y_{K\nu}(\Omega')\ S_{-1}(r, r') \tag{8.15}$$

with

$$S_{-1}(r, r') = \frac{(rr')^K}{2^{K+n/2-1}\Gamma(K+n/2)}\ {}_0F_1\left(\begin{array}{c} - \\ K+n/2 \end{array};\ -\frac{(rr')^2}{4}\right) \tag{8.16}$$

and

$$(-1)^{(N-K)/2} R_{NK}(r) = \int_0^\infty S_{-1}(r, r')\ R_{NK}(r')\, (r')^{n-1}\, dr' \tag{8.17}$$

as a special case of Eqs. (7.3) and (7.6) of Ref. [40] together with the orthogonality property of hyperspherical harmonics. We leave further details to the reader.

9. Particular Solutions of The Nonlinear Schrödinger Equations

The method of solving the equation (2.1) is extended in [20] to the nonlinear Schrödinger equation of the form

$$i\frac{\partial\psi}{\partial t} = -a(t)\frac{\partial^2\psi}{\partial x^2} + b(t)\, x^2\psi - i\left(c(t)\, x\frac{\partial\psi}{\partial x} + d(t)\,\psi\right) + h(t)\, |\psi|^{2s}\,\psi, \qquad s \geq 0. \tag{9.1}$$

We elaborate first on two cases (2.2) and (2.3). A particular solution takes the form

$$\psi = \psi(x, t) = K_h(x, y, t) = \frac{e^{i\phi}}{\sqrt{\mu(t)}}\ e^{i\left(\alpha(t)x^2 + \beta(t)xy + \gamma(t)y^2 + \kappa(t)\right)}, \qquad \phi = \text{constant}, \tag{9.2}$$

where equations (2.5)–(2.7) hold and, in addition,

$$\frac{d\kappa}{dt} = -\frac{h(t)}{\mu^s(t)}, \qquad \kappa(t) = \kappa(0) - \int_0^t \frac{h(\tau)}{\mu^s(\tau)}\, d\tau, \tag{9.3}$$

provided that the integral converges.

In the first case (2.2), by the superposition principle, the general solution of the characteristic equation (2.11) has the form

$$\mu \ = \ c_1\mu_1(t) + c_2\mu_2(t) \tag{9.4}$$

$$
\begin{aligned}
&= \cos t\left(c_1\cosh t+c_2\sinh t\right)+\sin t\left(c_1\sinh t+c_2\cosh t\right)\\
&= \frac{c_1+c_2}{\sqrt{2}}\ e^t\sin\left(t+\frac{\pi}{4}\right)+\frac{c_1-c_2}{\sqrt{2}}\ e^{-t}\cos\left(t+\frac{\pi}{4}\right)
\end{aligned}
$$

with $\mu' = 2\cos t\left(c_1\sinh t+c_2\cosh t\right)$ and

$$\mu(0)=c_1, \qquad \mu'(0)=2c_2. \tag{9.5}$$

Then

$$\alpha(t) = \frac{\cos t\left(c_1\sinh t+c_2\cosh t\right)-\sin t\left(c_1\cosh t+c_2\sinh t\right)}{2\left(\cos t\left(c_1\cosh t+c_2\sinh t\right)+\sin t\left(c_1\sinh t+c_2\cosh t\right)\right)}, \tag{9.6}$$

$$\beta(t) = \frac{c_1\beta(0)}{\cos t\left(c_1\cosh t+c_2\sinh t\right)+\sin t\left(c_1\sinh t+c_2\cosh t\right)}, \tag{9.7}$$

and

$$\gamma(t)=\gamma(0)-\frac{c_1\beta^2(0)\left(\cos t\sinh t+\sin t\cosh t\right)}{2\left(\cos t\left(c_1\cosh t+c_2\sinh t\right)+\sin t\left(c_1\sinh t+c_2\cosh t\right)\right)} \tag{9.8}$$

as a result of elementary but somewhat tedious calculations. The first two equations follow directly from (2.5) and a constant multiple of the first equation (2.6), respectively. One should use

$$\frac{d\gamma}{dt}+a(t)\,\beta^2=0, \tag{9.9}$$

see [20], integration by parts as in (2.7), and an elementary integral

$$\int\frac{dt}{\left(c_1\sinh t+c_2\cosh t\right)^2}=\frac{\sinh t}{c_2\left(c_1\sinh t+c_2\cosh t\right)}+C \tag{9.10}$$

in order to derive (9.8).

Two special cases are as follows. The original propagator (1.5) appears in the limit $c_1\to 0$ when $\beta(0)=-\left(c_1\right)^{-1}$ and $\gamma(0)=\left(2c_1c_2\right)^{-1}$. The solution with the standing wave initial data $\psi(x,0)=e^{ixy}$ found in [20] corresponds to $c_1=1$ and $c_2=0$.

Equation (9.3) can be explicitly integrated in some special cases, say, when $h(t)=\lambda\mu'(t)$:

$$\kappa(t)=\begin{cases}\kappa(0)-\dfrac{\lambda}{1-s}\left(\mu^{1-s}(t)-\mu^{1-s}(0)\right), & \text{when } s\neq 1,\\[2ex] \kappa(0)-\lambda\ln\left(\dfrac{\mu(t)}{\mu(0)}\right), & \text{when } s=1.\end{cases} \tag{9.11}$$

Here $\mu(0)\neq 0$; cf. [20]. One may treat the general particular solution of the form (9.2) with the coefficients (9.6)–(9.8) and (9.11) as an example of application of yet unknown "nonlinear" superposition principle for the Schrödinger equation under consideration for two particular solutions of a similar form with $c_1\neq 0$, $c_2=0$ and $c_1=0$, $c_2\neq 0$.

It is worth noting that function (9.2) with the coefficients given by (9.4)–(9.11) does also satisfy the following linear Schrödinger equation

$$i\frac{\partial\psi}{\partial t}=-a(t)\frac{\partial^2\psi}{\partial x^2}+b(t)\,x^2\psi-i\left(c(t)\,x\frac{\partial\psi}{\partial x}+d(t)\,\psi\right)+\frac{h(t)}{\mu^s(t)}\ \psi, \qquad s\geq 0. \tag{9.12}$$

Then a more general solution of this equation can be obtained by the superposition principle as follows

$$\psi(x,t)=\int_{-\infty}^{\infty}K_h(x,y,t)\ \chi(y)\ dy, \tag{9.13}$$

where χ is an arbitrary function such that the integral converges and one can interchange differentiation and integration. Solution of the Cauchy initial value problem simply requires an inversion of the integral

$$\psi(x,0)=\int_{-\infty}^{\infty}K_h(x,y,0)\ \chi(y)\ dy \tag{9.14}$$

that is

$$\chi(y)=\frac{c_1\beta(0)}{2\pi}\int_{-\infty}^{\infty}K_h^*(x,y,0)\ \psi(x,0)\ dx, \tag{9.15}$$

say, by the inverse of the Fourier transform. Thus our equations (9.13) and (9.15) solve the initial value problem for the above linear Schrödinger equation (9.12) as a double integral with the help of the kernel $K_h(x,y,t)$ that is regular at $t=0$, when $\mu(0)=c_1\neq 0$.

On the other hand,

$$K_h(x,y,t)=\int_{-\infty}^{\infty}G_h(x,z,t)\ K_h(z,y,0)\ dz, \tag{9.16}$$

where $G_h(x,y,t)$ is the Green function, which can be obtain from our solution (9.2) in the limit $c_1\to 0$ with a proper normalization as in the propagator (1.5). Therefore, substitution of (9.15) into (9.14) gives the traditional single integral form of the solution in terms of the Green function

$$\psi(x,t)=\int_{-\infty}^{\infty}G_h(x,y,t)\ \psi(y,0)\ dy \tag{9.17}$$

by (9.16).

The case of the new Hamiltonian, corresponding to (2.3), is similar. The general solution of characteristic equation (2.14) is given by

$$\begin{aligned}\mu &= c_2\mu_2(t)+c_3\mu_3(t) \\ &= \cos t\,(c_2\sinh t-c_3\cosh t)+\sin t\,(c_2\cosh t+c_3\sinh t) \\ &= \frac{1}{\sqrt{2}}\,e^{t}\left(c_2\sin\left(t+\frac{\pi}{4}\right)-c_3\cos\left(t+\frac{\pi}{4}\right)\right) \\ &\quad -\frac{1}{\sqrt{2}}\,e^{-t}\left(c_2\cos\left(t+\frac{\pi}{4}\right)+c_3\sin\left(t+\frac{\pi}{4}\right)\right)\end{aligned} \tag{9.18}$$

and $\mu'=2\cosh t\,(c_2\cos t+c_3\sin t)$ with $\mu(0)=-c_3$, $\mu'(0)=2c_2$. The first three coefficients of the quadratic form in the solution (9.2) are

$$\alpha(t) = \frac{\cos t\,(c_2\cosh t-c_3\sinh t)+\sin t\,(c_2\sinh t+c_3\cosh t)}{2\left(\cos t\,(c_2\sinh t-c_3\cosh t)+\sin t\,(c_2\cosh t+c_3\sinh t)\right)}, \tag{9.19}$$

$$\beta(t) = \frac{-c_3\beta(0)}{\cos t\,(c_2\sinh t-c_3\cosh t)+\sin t\,(c_2\cosh t+c_3\sinh t)}, \tag{9.20}$$

$$\gamma(t) = \gamma(0)+\frac{c_3\beta^2(0)\,(\cos t\sinh t+\sin t\cosh t)}{2\left(\cos t\,(c_2\sinh t-c_3\cosh t)+\sin t\,(c_2\cosh t+c_3\sinh t)\right)} \tag{9.21}$$

and one can use formula (9.11) for the last coefficient. The corresponding elementary integral is

$$\int \frac{dt}{\left(A\cos t+B\sin t\right)^{2}}=\frac{\sin t}{A\left(A\cos t+B\sin t\right)}+C. \tag{9.22}$$

The cases (2.18) and (2.19) can be considered in a similar fashion. The results are

$$\begin{aligned} \mu &= c_3\mu_3\left(t\right)+c_4\mu_4\left(t\right) \\ &= \sin t\left(c_3\sinh t+c_4\cosh t\right)-\cos t\left(c_3\cosh t+c_4\sinh t\right), \end{aligned} \tag{9.23}$$

$$\alpha\left(t\right) = \frac{\sin t\left(c_3\cosh t+c_4\sinh t\right)+\cos t\left(c_3\sinh t+c_4\cosh t\right)}{2\left(\sin t\left(c_3\sinh t+c_4\cosh t\right)-\cos t\left(c_3\cosh t+c_4\sinh t\right)\right)}, \tag{9.24}$$

$$\beta\left(t\right) = \frac{-c_3\beta\left(0\right)}{\sin t\left(c_3\sinh t+c_4\cosh t\right)-\cos t\left(c_3\cosh t+c_4\sinh t\right)}, \tag{9.25}$$

$$\gamma\left(t\right) = \gamma\left(0\right)+\frac{c_3\beta^2\left(0\right)\left(\sin t\cosh t-\cos t\sinh t\right)}{2\left(\sin t\left(c_3\sinh t+c_4\cosh t\right)-\cos t\left(c_3\cosh t+c_4\sinh t\right)\right)} \tag{9.26}$$

and

$$\begin{aligned} \mu &= c_1\mu_1\left(t\right)+c_4\mu_4\left(t\right) \\ &= \cos t\left(c_1\cosh t-c_4\sinh t\right)+\sin t\left(c_1\sinh t+c_4\cosh t\right), \end{aligned} \tag{9.27}$$

$$\alpha\left(t\right) = -\frac{\sinh t\left(c_1\cos t+c_4\sin t\right)+\cosh t\left(c_1\sin t-c_4\cos t\right)}{2\left(\sinh t\left(c_1\sin t-c_4\cos t\right)+\cosh t\left(c_1\cos t+c_4\sin t\right)\right)}, \tag{9.28}$$

$$\beta\left(t\right) = \frac{c_1\beta\left(0\right)}{\sinh t\left(c_1\sin t-c_4\cos t\right)+\cosh t\left(c_1\cos t+c_4\sin t\right)}, \tag{9.29}$$

$$\gamma\left(t\right) = \gamma\left(0\right)+\frac{c_1\beta^2\left(0\right)\left(\cos t\sinh t-\sin t\cosh t\right)}{2\left(\sinh t\left(c_1\sin t-c_4\cos t\right)+\cosh t\left(c_1\cos t+c_4\sin t\right)\right)}, \tag{9.30}$$

respectively. One can use once again formula (9.11) for the last coefficient. We leave further details to the reader.

10. A Note on The Ill-Posedness of The Schrödinger Equations

The same method shows that the joint solution of the both linear and nonlinear Schrödinger equations (9.12) and (9.1), respectively, corresponding to the initial data

$$\psi|_{t=0}=\delta_{\varepsilon}\left(x-y\right)=\frac{1}{\sqrt{2\pi i\varepsilon}}\ \exp\left(\frac{i\left(x-y\right)^{2}}{2\varepsilon}\right),\quad \varepsilon>0, \tag{10.1}$$

has the form

$$\psi=G_{\varepsilon}\left(x,y,t\right)=\frac{1}{\sqrt{i\mu_{\varepsilon}\left(t\right)}}\ e^{i\left(\alpha_{\varepsilon}(t)x^{2}+\beta_{\varepsilon}(t)xy+\gamma_{\varepsilon}(t)y^{2}+\kappa_{\varepsilon}(t)\right)} \tag{10.2}$$

with the characteristic function $\mu_{\varepsilon}\left(t\right)=2\pi\left(\varepsilon\mu_1\left(t\right)+\mu_2\left(t\right)\right).$ The coefficients of the quadratic form are given by

$$\alpha_{\varepsilon}\left(t\right) = \frac{\cos t\left(\varepsilon\sinh t+\cosh t\right)-\sin t\left(\varepsilon\cosh t+\sinh t\right)}{2\left(\cos t\left(\varepsilon\cosh t+\sinh t\right)+\sin t\left(\varepsilon\sinh t+\cosh t\right)\right)}, \tag{10.3}$$

$$\beta_{\varepsilon}(t) = -\frac{1}{\cos t(\varepsilon\cosh t+\sinh t)+\sin t(\varepsilon\sinh t+\cosh t)}, \tag{10.4}$$

$$\gamma_{\varepsilon}(t) = \frac{\cos t\cosh t+\sin t\sinh t}{2(\cos t(\varepsilon\cosh t+\sinh t)+\sin t(\varepsilon\sinh t+\cosh t))}. \tag{10.5}$$

We simply choose $c_1 = 2\pi\varepsilon > 0$, $c_2 = 2\pi$ and $e^{i\varphi} = 1/\sqrt{i}$ and the initial data $\alpha(0) = \gamma(0) = -\beta(0)/2 = 1/(2\varepsilon)$ in a general solution (9.6)–(9.8). The case $\varepsilon = 0$, $t > 0$ corresponds to the original propagator (1.5), while $\varepsilon > 0$, $t = 0$ gives the delta sequence (10.1).

If $h = h_{\varepsilon}(t) = (\lambda/2\pi)\mu_{\varepsilon}' = 2\lambda\cos t(\varepsilon\sinh t+\cosh t)$, then

$$\kappa_{\varepsilon}(t) = \begin{cases} -\dfrac{\lambda}{(2\pi)^s}\dfrac{(\varepsilon\mu_1(t)+\mu_2(t))^{1-s}-\varepsilon^{1-s}}{1-s}, & \text{when } 0\le s<1, \\ -\dfrac{\lambda}{2\pi}\ln\left(\mu_1(t)+\dfrac{\mu_2(t)}{\varepsilon}\right), & \text{when } s=1 \end{cases} \tag{10.6}$$

with $\kappa_{\varepsilon}(0) = 0$ provided $\varepsilon > 0$.

In this example, the initial data $\psi|_{t=0} = G_{\varepsilon}(x,y,0) = \delta_{\varepsilon}(x-y)$ converge to the Dirac delta function $\delta(x-y)$ as $\varepsilon\to 0^+$ in the distributional sense [17], [58], [64], [66]

$$\lim_{\varepsilon\to 0^+}\int_{-\infty}^{\infty} G_{\varepsilon}(x,y,0)\ \varphi(y)\ dy = \varphi(x). \tag{10.7}$$

On the other hand,

$$\begin{aligned} &\lim_{\varepsilon\to 0^+}\int_{-\infty}^{\infty} G_{\varepsilon}(x,y,t)\ \varphi(y)\ dy \\ &\quad = e^{i\lim_{\varepsilon\to 0^+}\kappa_{\varepsilon}(t)}\int_{-\infty}^{\infty} G_0(x,y,t)\ \varphi(y)\ dy \end{aligned} \tag{10.8}$$

with $t > 0$. When $s = 1$ the solution $\psi = G_{\varepsilon}(x,y,t)$, $t > 0$ does not have a limit because of divergence of the logarithmic phase factor $\kappa_{\varepsilon}(t)$ as $\varepsilon\to 0^+$. See also Refs. [3] and [33] on the ill-posedness of some canonical dispersive equations.

The second case, corresponding to (2.3), is similar. One can choose $\mu_{\varepsilon}(t) = 2\pi(\mu_2(t)-\varepsilon\mu_3(t))$ and obtain

$$\alpha_{\varepsilon}(t) = \frac{\cos t(\cosh t+\varepsilon\sinh t)+\sin t(\sinh t-\varepsilon\cosh t)}{2(\cos t(\sinh t+\varepsilon\cosh t)+\sin t(\cosh t-\varepsilon\sinh t))}, \tag{10.9}$$

$$\beta_{\varepsilon}(t) = -\frac{1}{\cos t(\sinh t+\varepsilon\cosh t)+\sin t(\cosh t-\varepsilon\sinh t)}, \tag{10.10}$$

$$\gamma_{\varepsilon}(t) = \frac{\cos t\cosh t-\sin t\sinh t}{2(\cos t(\sinh t+\varepsilon\cosh t)+\sin t(\cosh t-\varepsilon\sinh t))}. \tag{10.11}$$

If $h_{\varepsilon}(t) = (\lambda/2\pi)\mu_{\varepsilon}' = 2\lambda\cosh t(\cos t-\varepsilon\sin t)$, then

$$\kappa_{\varepsilon}(t) = \begin{cases} -\dfrac{\lambda}{(2\pi)^s}\dfrac{(\mu_2(t)-\varepsilon\mu_3(t))^{1-s}-\varepsilon^{1-s}}{1-s}, & \text{when } 0\le s<1, \\ -\dfrac{\lambda}{2\pi}\ln\left(\dfrac{\mu_2(t)}{\varepsilon}-\mu_3(t)\right), & \text{when } s=1 \end{cases} \tag{10.12}$$

and $\kappa_{\varepsilon}(0)=0$ when $\varepsilon>0$. Formulas (10.2) and (10.9)–(10.12) describe actual (nonlinear) evolution for initial data as in (10.1). One can observe once again a discontinuity with respect to these initial data as $\varepsilon \rightarrow 0^{+}$.

The cases (2.18) and (2.19) are as follows. One gets $\mu_{\varepsilon}(t)=2\pi\left(\mu_{4}(t)-\varepsilon\mu_{3}(t)\right)$,

$$
\begin{aligned}
\alpha_{\varepsilon}(t) &= \frac{\sin t\left(\sinh t-\varepsilon\cosh t\right)+\cos t\left(\cosh t-\varepsilon\sinh t\right)}{2\left(\sin t\left(\cosh t-\varepsilon\sinh t\right)-\cos t\left(\sinh t-\varepsilon\cosh t\right)\right)}, && (10.13)\\
\beta_{\varepsilon}(t) &= -\frac{1}{\sin t\left(\cosh t-\varepsilon\sinh t\right)-\cos t\left(\sinh t-\varepsilon\cosh t\right)}, && (10.14)\\
\gamma_{\varepsilon}(t) &= \frac{\cos t\cosh t-\sin t\sinh t}{2\left(\sin t\left(\cosh t-\varepsilon\sinh t\right)-\cos t\left(\sinh t-\varepsilon\cosh t\right)\right)} && (10.15)
\end{aligned}
$$

and $h_{\varepsilon}(t)=(\lambda/2\pi)\,\mu_{\varepsilon}'=2\lambda\sin t\left(\sinh t-\varepsilon\cosh t\right)$,

$$
\kappa_{\varepsilon}(t)=\begin{cases} -\dfrac{\lambda}{(2\pi)^{s}}\dfrac{\left(\mu_{4}(t)-\varepsilon\mu_{3}(t)\right)^{1-s}-\varepsilon^{1-s}}{1-s}, & \text{when } 0\leq s<1,\\[2ex] -\dfrac{\lambda}{2\pi}\ln\left(\dfrac{\mu_{4}(t)}{\varepsilon}-\mu_{3}(t)\right), & \text{when } s=1 \end{cases} \tag{10.16}
$$

with $\kappa_{\varepsilon}(0)=0$, $\varepsilon>0$ in the case (2.18). Also $\mu_{\varepsilon}(t)=2\pi\left(\mu_{4}(t)+\varepsilon\mu_{1}(t)\right)$,

$$
\begin{aligned}
\alpha_{\varepsilon}(t) &= \frac{\sinh t\left(\sin t+\varepsilon\cos t\right)-\cosh t\left(\cos t-\varepsilon\sin t\right)}{2\left(\sinh t\left(\cos t-\varepsilon\sin t\right)-\cosh t\left(\sin t+\varepsilon\cos t\right)\right)}, && (10.17)\\
\beta_{\varepsilon}(t) &= \frac{1}{\sinh t\left(\cos t-\varepsilon\sin t\right)-\cosh t\left(\sin t+\varepsilon\cos t\right)}, && (10.18)\\
\gamma_{\varepsilon}(t) &= -\frac{\cos t\cosh t+\sin t\sinh t}{2\left(\sinh t\left(\cos t-\varepsilon\sin t\right)-\cosh t\left(\sin t+\varepsilon\cos t\right)\right)} && (10.19)
\end{aligned}
$$

and $h_{\varepsilon}(t)=(\lambda/2\pi)\,\mu_{\varepsilon}'=2\lambda\sinh t\left(\sin t+\varepsilon\cos t\right)$,

$$
\kappa_{\varepsilon}(t)=\begin{cases} -\dfrac{\lambda}{(2\pi)^{s}}\dfrac{\left(\mu_{4}(t)+\varepsilon\mu_{1}(t)\right)^{1-s}-\varepsilon^{1-s}}{1-s}, & \text{when } 0\leq s<1,\\[2ex] -\dfrac{\lambda}{2\pi}\ln\left(\dfrac{\mu_{4}(t)}{\varepsilon}+\mu_{1}(t)\right), & \text{when } s=1 \end{cases} \tag{10.20}
$$

with $\kappa_{\varepsilon}(0)=0$, $\varepsilon>0$ in the case (2.19). We leave the details to the reader.

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11. Appendix A. Fundamental Solutions of The Characteristic Equations

We denote

$$u_1 = \cos t, \qquad u_2 = \sin t, \qquad v_1 = \cosh t, \qquad v_2 = \sinh t \tag{11.1}$$

such that $u_1' = -u_2$, $u_2' = u_1$, $v_1' = v_2$, $v_2' = v_1$ and study differential equations satisfied by the following set of the Wronskians of trigonometric and hyperbolic functions

$$\{W(u_\alpha, v_\beta)\}_{\alpha,\beta=1,2} = \{W(u_1,v_1),\ W(u_1,v_2),\ W(u_2,v_1),\ W(u_2,v_2)\}. \tag{11.2}$$

Let us take, for example,

$$y = W(u_1,v_1) = u_1v_2 + u_2v_1. \tag{11.3}$$

Then

$$y' = 2u_1v_1, \qquad y'' = 2u_1v_2 - 2u_2v_1 \tag{11.4}$$

and

$$y'' - \tau y' + 4\sigma y = (4\sigma+2)\,u_1v_2 + (4\sigma-2)\,u_2v_1 - 2\tau u_1v_1 = 0. \tag{11.5}$$

The last equation is satisfied when $\sigma = 1/2$, $\tau = 2v_2/v_1$ and $\sigma = -1/2$, $\tau = -2u_2/u_1$. All other cases are similar and the results are presented in Table 1.

Our calculations reveal the following identities

$$\begin{aligned} W''(u_1,v_1) &= -2W(u_2,v_2), \qquad & W''(u_1,v_2) &= -2W(u_2,v_1), \\ W''(u_2,v_1) &= 2W(u_1,v_2), \qquad & W''(u_2,v_2) &= 2W(u_1,v_1), \end{aligned} \tag{11.6}$$

for the Wronskians under consideration. This implies that the set of Wronskians (11.2) provides the fundamental solutions of the fourth order differential equation

$$W^{(4)} + 4W = 0 \tag{11.7}$$

with constant coefficients. The corresponding characteristic equation, $\lambda^4 + 4 = 0$, has four roots, $\lambda_1 = 1+i$, $\lambda_2 = 1-i$, $\lambda_3 = -1+i$, $\lambda_4 = -1-i$, and the fundamental solution set is given by

$$\{u_\alpha v_\beta\}_{\alpha,\beta=1,2} = \{u_1v_1,\ u_1v_2,\ u_2v_1,\ u_2v_2\}. \tag{11.8}$$

These solutions of the bi-harmonic equation (11.7) are even or odd functions of time. They do not satisfy our second order characteristic equations. For example, let $w_1 = u_1v_2 = \cos t \sinh t$ and $w_1 = u_2v_1 = \sin t \cosh t$. Then, by a direct calculation,

$$\begin{aligned} L(w_1) &= w_1'' + 2\tan t\ w_1' - 2w_1 = -2\frac{\sinh t}{\cos t}, \\ L(w_2) &= w_2'' + 2\tan t\ w_2' - 2w_2 = 2\frac{\sinh t}{\cos t}. \end{aligned} \tag{11.9}$$

Thus, separately, these solutions of (11.7) satisfy nonhomogeneous characteristic equations. But together,

$$L(y_1) = L(w_1+w_2) = L(w_1) + L(w_2) = -2\frac{\sinh t}{\cos t} + 2\frac{\sinh t}{\cos t} = 0. \tag{11.10}$$

A similar property holds for all other solutions of the characteristic equations from Table 1.

Table 1. Fundamental solutions of the characteristic equations.

Characteristic equation $y'' - \tau y' + 4\sigma y = 0$	Fundamental solution set $\{y_i, y_k\}_{i<k}$
$u'' + u = 0$ $(\sigma = 1/4, \quad \tau = 0)$	$u_1 = \cos t, \qquad u_2 = \sin t$ $(u_1' = -u_2, \quad u_2' = u_1)$
$v'' - v = 0$ $(\sigma = -1/4, \quad \tau = 0)$	$v_1 = \cosh t, \qquad v_2 = \sinh t$ $(v_1' = v_2, \quad v_2' = v_1)$
$y'' + 2\tan t\ y' - 2y = 0$ $(\sigma = -1/2, \quad \tau = -2u_2/u_1)$	$y_1 = W(u_1, v_1) = u_1v_2 + u_2v_1$ $y_2 = W(u_1, v_2) = u_1v_1 + u_2v_2$
$y'' - 2\cot t\ y' - 2y = 0$ $(\sigma = -1/2, \quad \tau = 2u_2/u_1)$	$y_3 = W(u_2, v_2) = u_2v_1 - u_1v_2$ $y_4 = W(u_2, v_1) = u_2v_2 - u_1v_1$
$y'' - 2\tanh t\ y' + 2y = 0$ $(\sigma = 1/2, \quad \tau = 2v_2/v_1)$	$y_1 = W(u_1, v_1) = u_1v_2 + u_2v_1$ $y_4 = W(u_2, v_1) = u_2v_2 - u_1v_1$
$y'' - 2\coth t\ y' + 2y = 0$ $(\sigma = 1/2, \quad \tau = 2v_1/v_2)$	$y_2 = W(u_1, v_2) = u_1v_1 + u_2v_2$ $y_3 = W(u_2, v_2) = u_2v_1 - u_1v_2$

In order to obtain the fundamental solutions in an algebraic manner, we denote

$$L_1 = \frac{d^2}{dt^2} + 2\frac{u_2}{u_1}\frac{d}{dt} - 2, \quad L_2 = \frac{d^2}{dt^2} - 2\frac{u_1}{u_2}\frac{d}{dt} - 2, \tag{11.11}$$
$$L_3 = \frac{d^2}{dt^2} - 2\frac{v_2}{v_1}\frac{d}{dt} + 2, \quad L_4 = \frac{d^2}{dt^2} - 2\frac{v_1}{v_2}\frac{d}{dt} + 2$$

and compute the actions of these second order linear differential operators L_k on the four basis vectors $\{u_\alpha v_\beta\}_{\alpha,\beta=1,2}$, namely, $L_k(u_\alpha v_\beta)$. The results are presented in Table 2.

Table 2. Construction of the fundamental solutions.

Linear operators	u_1v_1	u_1v_2	u_2v_1	u_2v_2
$\frac{d}{dt}$	$u_1v_2 - u_2v_1$	$u_1v_1 - u_2v_2$	$u_1v_1 + u_2v_2$	$u_1v_2 + u_2v_1$
$\frac{d^2}{dt^2}$	$-2u_2v_2$	$-2u_2v_1$	$2u_1v_2$	$2u_1v_1$
L_1	$-2\frac{v_1}{u_1}$	$-2\frac{v_2}{u_1}$	$2\frac{v_2}{u_1}$	$2\frac{v_1}{u_1}$
L_2	$-2\frac{v_2}{u_2}$	$-2\frac{v_1}{u_2}$	$-2\frac{v_1}{u_2}$	$-2\frac{v_2}{u_2}$
L_3	$2\frac{u_1}{v_1}$	$-2\frac{u_2}{v_1}$	$2\frac{u_2}{v_1}$	$2\frac{u_1}{v_1}$
L_4	$2\frac{u_2}{v_2}$	$-2\frac{u_1}{v_2}$	$-2\frac{u_1}{v_2}$	$-2\frac{u_2}{v_2}$

Therefore

$$\begin{aligned} L_1(u_1v_1 + u_2v_2) &= L_1(u_1v_2 + u_2v_1) \\ &= L_2(u_2v_2 - u_1v_1) = L_2(u_2v_1 - u_1v_2) \\ &= L_3(u_1v_2 + u_2v_1) = L_3(u_2v_2 - u_1v_1) \\ &= L_4(u_1v_1 + u_2v_2) = L_4(u_2v_1 - u_1v_2) = 0 \end{aligned} \tag{11.12}$$

as has been stated in Table 1.

All our characteristic equations in this paper obey certain periodicity properties. For instance, equations

$$y'' + 2\tan t \; y' - 2y = 0 \tag{11.13}$$

and

$$y'' - 2\cot t \; y' - 2y = 0 \tag{11.14}$$

are invariant under the shifts $t \rightarrow t \pm \pi$ and interchange one into another when $t \rightarrow t \pm \pi/2$. Since only two solutions of a linear second order differential equation may be linearly independent, the corresponding fundamental solutions satisfy the following relations

$$\begin{pmatrix} y_1(t \pm \pi) \\ y_2(t \pm \pi) \end{pmatrix} = -\begin{pmatrix} \cosh \pi & \pm \sinh \pi \\ \pm \sinh \pi & \cosh \pi \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} \tag{11.15}$$

and

$$\begin{pmatrix} y_1(t \pm \pi/2) \\ y_2(t \pm \pi/2) \end{pmatrix} = -\begin{pmatrix} \sinh(\pi/2) & \pm \cosh(\pi/2) \\ \pm \cosh(\pi/2) & \sinh(\pi/2) \end{pmatrix} \begin{pmatrix} y_3(t) \\ y_4(t) \end{pmatrix}, \tag{11.16}$$

respectively. Two other characteristic equations have pure imaginary periods. We leave the details to the reader.

12. Appendix B. On A Transformation of The Quantum Hamiltonians

Our definition of the creation and annihilation operators given by (1.3) implies the following operator identities

$$x^2 = \frac{1}{2}\left(aa^{\dagger} + a^{\dagger}a\right) - \frac{1}{2}\left(a^2 + \left(a^{\dagger}\right)^2\right), \tag{12.1}$$

$$\frac{\partial^2}{\partial x^2} = -\frac{1}{2}\left(aa^{\dagger} + a^{\dagger}a\right) - \frac{1}{2}\left(a^2 + \left(a^{\dagger}\right)^2\right), \tag{12.2}$$

$$2x\frac{\partial}{\partial x} + 1 = -a^2 + \left(a^{\dagger}\right)^2 \tag{12.3}$$

(and vise versa), which allows us to transform the time-dependent Schrödinger equation (2.1) into a Hamiltonian form (1.1), where the Hamiltonian is written in terms of the creation and annihilation operators as follows

$$\begin{aligned} H &= \frac{1}{2}\left(a(t) + b(t)\right)\left(aa^{\dagger} + a^{\dagger}a\right) \\ &\quad + \frac{1}{2}\left(a(t) - b(t) + 2id(t)\right)a^2 + \frac{1}{2}\left(a(t) - b(t) - 2id(t)\right)\left(a^{\dagger}\right)^2, \end{aligned} \tag{12.4}$$

when $c = 2d$. This helps to transform the Hamiltonians of modified oscillators under consideration into different equivalent forms, which are used in the paper.

The trigonometric cases (2.2) and (2.18) results in the Hamiltonians (1.2) and (7.18) with $n = 1$, respectively. In the first hyperbolic case (2.3) one gets

$$H = \frac{1}{2}\cosh(2t)\left(aa^{\dagger} + a^{\dagger}a\right) + \frac{1}{2}\left(1 - i\sinh(2t)\right)a^2 + \frac{1}{2}\left(1 + i\sinh(2t)\right)\left(a^{\dagger}\right)^2, \tag{12.5}$$

where

$$1 \pm i\sinh(2t) = \cosh(2t)\, e^{\pm i \arctan(2\tau)}, \qquad \tau = \frac{1}{2}\sinh(2t), \tag{12.6}$$

which implies the Schrödinger equation (1.6)–(1.7). The second hyperbolic case (2.19) is similar. We leave the details to the reader.

13. Appendix C. On A Hamiltonian Structure of The Characteristic Equations

The Hamilton equations of classical mechanics [35],

$$\dot{q}=\frac{\partial H}{\partial p}, \qquad \dot{p}=-\frac{\partial H}{\partial q}, \tag{13.1}$$

with a general quadratic Hamiltonian

$$H=a\left(t\right)p^{2}+b\left(t\right)q^{2}+2d\left(t\right)pq \tag{13.2}$$

are

$$\dot{q}=2ap+2dq, \qquad \dot{p}=-2bq-2dp. \tag{13.3}$$

We denote, as is customary, differentiation with respect to time by placing a dot above the canonical variables p and q. Elimination of the generalized momentum p from this system results in the second order equation with respect to the generalized coordinate

$$\ddot{q}-\frac{\dot{a}}{a}\,\dot{q}+4\left(ab-d^{2}+\frac{d}{2}\left(\frac{\dot{a}}{a}-\frac{\dot{d}}{d}\right)\right)q=0. \tag{13.4}$$

It coincides with the characteristic equation (2.8)–(2.9) with $c=2d$. Our choice of the coefficients (2.2)–(2.3) and (2.18)–(2.19) in the classical Hamiltonian (13.2) corresponds to the following models of modified classical oscillators

$$\begin{aligned}
\ddot{q}+2\tan t\;\dot{q}-2q &= 0, && (13.5)\\
\ddot{q}-2\tanh t\;\dot{q}+2q &= 0, && (13.6)\\
\ddot{q}-2\cot t\;\dot{q}-2q &= 0, && (13.7)\\
\ddot{q}-2\coth t\;\dot{q}+2q &= 0, && (13.8)
\end{aligned}$$

respectively; see Appendix A for their fundamental solutions.

The standard quantization of the classical integrable systems under consideration, namely,

$$q\rightarrow x, \qquad p\rightarrow i^{-1}\frac{\partial}{\partial x}, \qquad [x,p]=xp-px=i \tag{13.9}$$

and

$$H\rightarrow ap^{2}+bx^{2}+d\left(px+xp\right), \qquad i\frac{\partial\psi}{\partial t}=H\psi, \tag{13.10}$$

leads to the quantum exactly solvable models of modified oscillators discussed in this paper.

Another example is a damped oscillator with $a=\left(\omega_{0}/2\right)e^{-2\lambda t}$, $b=\left(\omega_{0}/2\right)e^{2\lambda t}$ and $c=d=0$. The classical equation

$$\ddot{q}+2\lambda\;\dot{q}+\omega_{0}^{2}\;q=0 \tag{13.11}$$

describes damped oscillations [35]. The corresponding quantum propagator has the form (2.4) with

$$\mu=\frac{\omega_{0}}{\omega}e^{-\lambda t}\sin\omega t, \qquad \omega^{2}=\omega_{0}^{2}-\lambda^{2}>0 \tag{13.12}$$

and

$$\alpha(t) = \frac{\omega\cos\omega t - \lambda\sin\omega t}{2\omega_0\sin\omega t}e^{2\lambda t}, \tag{13.13}$$

$$\beta(t) = -\frac{\omega}{\omega_0\sin\omega t}e^{\lambda t}, \tag{13.14}$$

$$\gamma(t) = \frac{\omega^2 - \omega_0^2\sin^2\omega t}{2\omega_0\sin\omega t\,(\omega\cos\omega t - \lambda\sin\omega t)}. \tag{13.15}$$

The Schrödinger equation

$$i\frac{\partial\psi}{\partial t} = \frac{\omega_0}{2}\left(-e^{-2\lambda t}\frac{\partial^2\psi}{\partial x^2} + e^{2\lambda t}x^2\psi\right) \tag{13.16}$$

describes the linear oscillator with a variable unit of length $x \to xe^{\lambda t}$. See [21] for more details.

References

[1] G. E. Andrews, R. A. Askey, and R. Roy, *Special Functions*, Cambridge University Press, Cambridge, 1999.

[2] G. P. Arrighini and N. L. Durante, *More on the quantum propagator of a particle in a linear potential*, Am. J. Phys. **64** (1996) # 8, 1036–1041.

[3] V. Banica and L. Vega, *On the Dirac delta as initial condition for nonlinear Schrödinger equations*, arXiv:0706.0819v1 [math.AP] 6 Jun 2007.

[4] V. Bargmann, *Irreducible unitary representations of the Lorentz group*, Annals of Mathematics (2) **48** (1947), 568–640.

[5] L. A. Beauregard, *Propagators in nonrelativistic quantum mechanics*, Am. J. Phys. **34** (1966), 324–332.

[6] N. N. Bogoliubov and D. V. Shirkov, *Introduction to the Theory of Quantized Fields*, third edition, John Wiley & Sons, New York, Chichester, Brisbane, Toronto, 1980.

[7] L. M. A. Bettencourt, A. Cintrón-Arias, D. I. Kaiser, and C. Castillo-Chávez, *The power of a good idea: Quantitative modeling of the spread of ideas from epidemiological models*, Phisica A **364** (2006), 513–536.

[8] L. S. Brown and Y. Zhang, *Path integral for the motion of a particle in a linear potential*, Am. J. Phys. **62** (1994) # 9, 806–808.

[9] J. R. Cannon, *The One-Dimensional Heat Equation*, Encyclopedia of Mathematics and Its Applications, Vol. 32, Addison–Wesley Publishing Company, Reading etc, 1984.

[10] R. Carles, *Semi-classical Schrödinger equations with harmonic potential and nonlinear perturbation*, Ann. Henri Poincaré **3** (2002) #4, 757–772.

[11] R. Carles, *Remarks on nonlinear Schrödinger equations with harmonic potential*, Ann. Inst. H. Poincaré Anal. Non Linéaire **20** (2003) #3, 501–542.

[12] R. Carles, *Nonlinear Schrödinger equations with repulsive harmonic potential and applications*, SIAM J. Math. Anal. **35** (2003) #4, 823–843.

[13] R. Carles, *Global existence results for nonlinear Schrödinger equations with quadratic potentials*, arXiv: math/0405197v2 [math.AP] 18 Feb 2005.

[14] R. Carles, *Semi-Classical Analysis for Nonlinear Schrödinger Equations*, World Scientific, New Jersey etc, 2008.

[15] R. Carles and L. Miller, *Semiclassical nonlinear Schrödinger equations with potential and focusing initial data*, Osaka J. Math. **41** (2004) #3, 693–725.

[16] R. Carles and Y. Nakamura, *Nonlinear Schrödinger equations with Stark potential*, Hokkaido Math. J. **33** (2004) #3, 719–729.

[17] L. Carleson, *Some Analytic Problems Related to Statistical Mechanics*, in: *"Euclidean Harmonic Analysis"*, Lecture Notes in Math., Vol. 779, Springer–Verlag, Berlin, 1980, pp. 5–45.

[18] T. Cazenave, *Semilinear Schrödinger Equations*, Courant Lecture Notes in Mathematics, Vol. 10, American Mathematical Society, Providence, Rhode Island, 2003.

[19] T. Cazenave and A. Haraux, *An Introduction to Semilinear Evolution Equations*, Oxford Lecture Series in Mathematics and Its Applications, Vol. 13, Oxford Science Publications, Claredon Press, Oxford, 1998.

[20] R. Cordero-Soto, R. M. Lopez, E. Suazo, and S. K. Suslov, *Propagator of a charged particle with a spin in uniform magnetic and perpendicular electric fields*, Lett. Math. Phys. **84** (2008) #2–3, 159–178.

[21] R. Cordero-Soto, E. Suazo, and S. K. Suslov, *Models of damped oscillators in quantum mechanics*, in preparation.

[22] L. D. Faddeev and L. A. Takhtajan, *Hamiltonian Methods in the Theory of Solitons*, Springer–Verlag, Berlin, New York, 1987.

[23] R. P. Feynman, *The Principle of Least Action in Quantum Mechanics*, Ph. D. thesis, Princeton University, 1942; reprinted in: *"Feynman's Thesis – A New Approach to Quantum Theory"*, (L. M. Brown, Editor), World Scientific Publishers, Singapore, 2005, pp. 1–69.

[24] R. P. Feynman, *Space-time approach to non-relativistic quantum mechanics*, Rev. Mod. Phys. **20** (1948) # 2, 367–387; reprinted in: *"Feynman's Thesis – A New Approach to Quantum Theory"*, (L. M. Brown, Editor), World Scientific Publishers, Singapore, 2005, pp. 71–112.

[25] R. P. Feynman, *The theory of positrons*, Phys. Rev. **76** (1949) # 6, 749–759.

[26] R. P. Feynman, *Space-time approach to quantum electrodynamics*, Phys. Rev. **76** (1949) # 6, 769–789.

[27] R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals*, McGraw–Hill, New York, 1965.

[28] S. Flügge, *Practical Quantum Mechanics*, Springer–Verlag, Berlin, 1999.

[29] K. Gottfried and T.-M. Yan, *Quantum Mechanics: Fundamentals*, second edition, Springer–Verlag, Berlin, New York, 2003.

[30] D. R. Haaheim and F. M. Stein, *Methods of solution of the Riccati differential equation*, Mathematics Magazine **42** (1969) #2, 233–240.

[31] B. R. Holstein, *The linear potential propagator*, Am. J. Phys. **65** (1997) #5, 414–418.

[32] B. R. Holstein, *The harmonic oscillator propagator*, Am. J. Phys. **67** (1998) #7, 583–589.

[33] C. E. Kenig, G. Ponce, and L. Vega, *On the ill-posedness of some canonical dispersive equations*, Duke Math. J. **106** (2001) #3, 617–633.

[34] O. A. Ladyženskaja, V. A. Solonnikov, and N. N. Ural'ceva, *Linear and Quasilinear Equations of Parabolic Type*, Translations of Mathematical Monographs, Vol. 23, American Mathematical Sociaty, Providence, Rhode Island, 1968. (pp. 318, 356).

[35] L. D. Landau and E. M. Lifshitz, *Mechanics*, Pergamon Press, Oxford, 1976.

[36] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics: Nonrelativistic Theory*, Pergamon Press, Oxford, 1977.

[37] R. M. Lopez and S. K. Suslov, *The Cauchy problem for a forced harmonic oscillator*, arXiv:0707.1902v8 [math-ph] 27 Dec 2007.

[38] W. Magnus and S. Winkler, *Hill's Equation*, Dover Publications, New York, 1966.

[39] V. P. Maslov and M. V. Fedoriuk, *Semiclassical Approximation in Quantum Mechanics*, Reidel, Dordrecht, Boston, 1981.

[40] M. Meiler, R. Cordero-Soto, and S. K. Suslov, *Solution of the Cauchy problem for a time-dependent Schrödinger equation*, J. Math. Phys. **49** (2008) #7, 072102: 1–27; published on line 9 July 2008, URL: http://link.aip.org/link/?JMP/49/072102; see also arXiv: 0711.0559v4 [math-ph] 5 Dec 2007.

[41] I. V. Melinikova and A. Filinkov, *Abstract Cauchy problems: Three Approaches*, Chapman&Hall/CRC, Boca Raton, London, New York, Washington, D. C., 2001.

[42] E. Merzbacher, *Quantum Mechanics*, third edition, John Wiley & Sons, New York, 1998.

[43] A. M. Molchanov, *The Riccati equation $y' = x + y^2$ for the Airy function*, [in Russian], Dokl. Akad. Nauk **383** (2002) #2, 175–178.

[44] V. Naibo and A. Stefanov, *On some Schrödinger and wave equations with time dependent potentials*, Math. Ann. **334** (2006) # 2, 325–338.

[45] P. Nardone, *Heisenberg picture in quantum mechanics and linear evolutionary systems*, Am. J. Phys. **61** (1993) # 3, 232–237.

[46] A. F. Nikiforov, S. K. Suslov, and V. B. Uvarov, *Classical Orthogonal Polynomials of a Discrete Variable*, Springer–Verlag, Berlin, New York, 1991.

[47] A. F. Nikiforov and V. B. Uvarov, *Special Functions of Mathematical Physics*, Birkhäuser, Basel, Boston, 1988.

[48] J. D. Paliouras and D. S. Meadows, *Complex Variables for Scientists and Engineers*, second edition, Macmillan Publishing Company, New York and London, 1990.

[49] E. D. Rainville, *Special Functions*, The Macmillan Company, New York, 1960.

[50] V. M. Pérez-Garcia, P. Torres, and G. D. Montesinos, *The method of moments for nonlinear Schrödinger equations: theory and applications*, SIAM J. Appl. Math. **67** (2007) #4, 990–1015.

[51] E. D. Rainville, *Intermediate Differential Equations*, Wiley, New York, 1964.
[52] S. S. Rajah and S. D. Maharaj, *A Riccati equation in radiative stellar collapse*, J. Math. Phys. **49** (2008) #1, published on line 23 January 2008.
[53] R. W. Robinett, *Quantum mechanical time-development operator for the uniformly accelerated particle*, Am. J. Phys. **64** (1996) #6, 803–808.
[54] I. Rodnianski and W. Schlag, *Time decay for solutions of Schrödinger equations with rough and time-dependent potentials*, Invent. Math. **155** (2004) # 3, 451–513.
[55] O. S. Rozanova, *Hydrodynamic approach to constructing solutions of nonlinear Schrödinger equations in the critical case*, Proc. Amer. Math. Soc. **133** (2005), 2347–2358.
[56] W. Rudin, *Principles of Mathematical Analysis*, third edition, McGraw–Hill, New York, 1976.
[57] W. Schlag, *Dispersive estimates for Schrödinger operators: a survay*, arXiv: math/0501037v3 [math.AP] 10 Jan 2005.
[58] P. Sjölin, *Regularity of solutions to the Schrödinger equation*, Duke Math. J. **55** (1987) # 3, 699–715.
[59] Yu. F. Smirnov and K. V. Shitikova, *The method of K harmonics and the shell model*, Soviet Journal of Particles & Nuclei **8** (1977) #4, 344–370.
[60] G. Staffilani and D. Tataru, *Strichartz estimates for a Schrödinger operator with nonsmooth coefficients*, Commun. in Partial Diff. Eqns. **27** (2002) #5&6, 1337–1372.
[61] E. Suazo, and S. K. Suslov, *An integral form of the nonlinear Schrödinger equation with wariable coefficients*, arXiv:0805.0633v2 [math-ph] 19 May 2008.
[62] E. Suazo, S. K. Suslov, and J. M. Vega, *The Riccati differential equation and a diffusion-type equation*, arXiv:0807.4349v4 [math-ph] 8 Aug 2008.
[63] S. K. Suslov and B. Trey, *The Hahn polynomials in the nonrelativistic and relativistic Coulomb problems*, J. Math. Phys. **49** (2008) #1, 012104: 1–51; published on line 22 January 2008, URL: http://link.aip.org/link/?JMP/49/012104.
[64] T. Tao, *Nonlinear Dispersive Equations: Local and Global Analysis*, CBMS regional conference series in mathematics, 2006.
[65] N. S. Thomber and E. F. Taylor, *Propagator for the simple harmonic oscillator*, Am. J. Phys. **66** (1998) # 11, 1022–1024.
[66] L. Vega, *Schrödinger equations: pointwise convergence to initial data*, Proc. Amer. Math. Soc. **102** (1988) # 4, 874–878.
[67] N. Ya. Vilenkin, *Special Functions and the Theory of Group Representations*, American Mathematical Society, Providence, 1968.
[68] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, Second Edition, Cambridge University Press, Cambridge, 1944.
[69] E. P. Wigner, *Group Theory and its Application to the Quantum Mechanics of Atomic Spectra*, Academic Press, New York, 1959.

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